Successive differentiation and Leibnitz's formula

Objectives

In this section you will learn the following:

- The notion of successive differentiation.
- The Leibnitz's formula
- The notion of related rates

Successive Differentiation

Definition:

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. We say f is twice differentiable at $c \in I$ if f is differentiable on $(c-\delta, c+\delta)$ for some $\delta > 0$ and the derivative function is differentiable at c. In that case we define the second order derivative of f at c to be $f' : (c-\delta, c+\delta) \to \mathbb{R}$

$$f''(c) := (f')'(c),$$

the derivative of the derivative function. It is also denoted by

$$\frac{d^2 f}{dx}\bigg|_{x=c}$$

The concept of n - times differentiability and the nth derivative of f at c, denoted by $f^{(n)}(c)$, can be defined similarly:

$$f^{(n)}(c) := (f^{(n-1)})'(c)$$

If $f^{(n)}(c)$ exists for every $n \in \mathbb{N}$, we say f is infinitely differentiable at c.

Examples:

(i) Consider $f(x) = x^k, x \in \mathbb{R}$

Then f is n - times differentiable for every and $n \geq 1$ and

$$f^{(n)}(x) = \begin{cases} k(k-1)\,..\,(k-n)\ x^{k-n} & \text{for } n \leq k, \\ 0 & \text{for } n > k, \end{cases}$$

(ii) Let $f(x) = Sin\ x, x \in \mathbb{R}$. Then f is also n - times differentiable for every n . It is easy to show that

$$f^{n}\left(x\right) = sin(x + n\frac{\pi}{2})$$

(iii) Let

$$f(x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Then f is differentiable at every x with

$$f'(x) = \begin{cases} 2x\,sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

For $f, f''(0)$ does not exist. In fact, $f'(x)$ is not even continuous at $x = 0$.

The product rule for differentiation: for differentiable functions f and g ,

$$\left(fg\right)' = f'g + fg'$$

can be extended to higher derivatives as follows.

Theorem (Leibnitz's Rule):

Then f and g is n - times differentiable at a point c and both have all derivatives of orders up to $(n-1)$ in a neighborhood of c. Then, fg is differentiable at c with

$$(fg)^{(n)}(c) = f^{(n)}(c)g(c) + \binom{n}{1} f^{(n-1)}(c)g'(c) + \ . \ . \ . \ + \binom{n}{k} f^{(n-k)}(c)g^{(k)}(c) + . \ . \ . + f(c)g^{(n)}(c)$$

Proof:

It is easy to prove the required statement by induction on n. We leave the details to the reader.

Example:

Let us use Leibnitz's rule to find the third derivative of the function $h(x) = x^2 sin\ x$

Let

$$f(x) = x^2,\ g(x) = sin\ x$$

Then

$$f'(x) = 2x,\ f^{(2)}(x) = 2,\ f^{(3)}(x) = 0$$

and

$$g'(x) = \ cos\ x,\ g^{(2)}(x) = -sin\ x,\ g^{(3)}(x) = -\ cos\ x .$$

Thus

$$\begin{aligned} h^{(3)}(x) &= 3\,f^{(2)}(x)\,g'(x) + 3\,f'(x)\,g^{(2)}(x) + f(x)\,g^{(3)}(x) \\ &= 6\ cos\ x - 6\ x\ sin\ x - x^2\ cos\ x. \\ &= (6 - x^2)\ cos\ x - 6x\ sin\ x. \end{aligned}$$

We saw that the derivative of a function also represents the rate of change of the function. This interpretation along with the chain rule is useful in solving problems which involve various rates of change.

Example:

If the length of a rectangle decreases at the rate of 3 cm/sec and its width increases at the rate of 2 cm/sec, find the rate of change of the area of the rectangle when its length is 10 cms and its width is 4cms. Let x denote length, y denote width and A denote the area of the rectangle. Then by implicit differentiation

$$A = xy \Rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + \frac{dx}{dt}y = 2x - 3y$$

In particular, $x = 10, y = 4 \Rightarrow \frac{dA}{dt} = 8 \text{ cm}^2/\text{sec}$ that is, the area of the rectangle increases at the rate of $8 \text{ cm}^2/\text{sec}$

Example:

An airplane is flying in a straight path at a height of 6 Km from the ground which passes directly above a man standing on the ground. The distance s of the man from the plane is decreasing at the rate of 400 km per hour when $s = 10 \text{ km}$. We want to find the speed of the plane. To find this, let x denote the horizontal distance of the plane from the man. We note that for $s = 10,\ x = \sqrt{(10)^2 - (6)^2} = 8$. We are given that

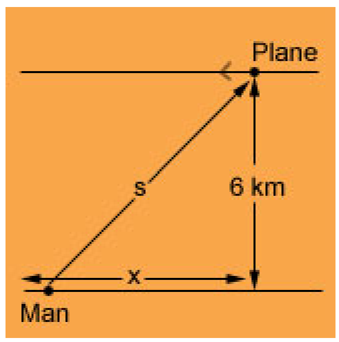
$$\frac{ds}{dt} = 400 \text{ when } s = 10$$,

and we have to find $\frac{dx}{dt}$. The variables s and x are related by $s^2 = x^2 + 6$. Thus,

$$2\frac{ds}{dt} = 2\frac{dx}{dt}$$

Hence

$$\frac{dx}{dt} = \left(\frac{s}{x}\right)\left(\frac{ds}{dt}\right)$$



Thus, when $s = 10,\ x = 8,\ \frac{ds}{dt} = -400$,
we set

$$\frac{dx}{dt} = \frac{10}{8}(-400) = -500 \text{ km/hour}$$

Hence the plane is approaching the man with a speed of 500 km/hour.

Practice Exercises: Successive differentiation

1. Let $f(x) = (x-c)^2 g(x)$, where $c \in \mathbb{R}$ and $g(x)$ and is differentiable and $g(c) \neq 0$ .Show that $f''(c) \neq 0$.

2. Show that for every $n \geq 1$,

(i) $\frac{d(\sin x)}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$

(ii) $\frac{d(\cos x)}{dx^n} = \cos\left(x + \frac{n\pi}{2}\right)$

3. Use Leibnitz theorem to find the third derivative of the functions

(i) $x^2 \sin x$

(ii) $x^2(x^2+1)^{-1}$

4. Using induction on n, show that

(i) For $f(x) = \frac{1}{1+x}, x \neq 1, f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$

(ii) For $f(x) = \frac{4}{x} - \frac{1}{x^2}, x \neq 0, f^{(n)}(x) = \frac{(-1)^n n!(3-n)}{x^{n+2}}$

5. The radius of the circular disc is increasing with time (think of oil pouring from a tanker in sec). Find the rate of change of the area of the disc to the radius of the disc. How fast is the area increasing when radius is 4km, if the rate of change of the radius is 5 cm/sec.