Rolle's Theorem and Cauchy's Mean Value Theorem

Objectives

In this section you will learn the following:

• Roll's theorem
• Mean Value Theorem
• Applications of Roll's Theorem

Rolle's Theorem

We saw in the previous lectures that continuity and differentiability help to understand some aspects of a function:

• Continuity of \( f \) tells us that its graph does not have any breaks.
• Differentiability of \( f \) tells us that its graph has no sharp edges.

In this section, we shall see how the knowledge about the derivative function \( f' \) help to understand the function \( f \) better.

Definitions:

Let \( A \subseteq \mathbb{R} \) and \( f : A \to \mathbb{R} \).

(i) We say \( f \) is increasing in \( A \) if

\[ x_1, x_2 \in A \quad \text{and} \quad x_1 < x_2 \quad \text{implies} \quad f(x_1) \leq f(x_2). \]

(ii) We say \( f \) is decreasing in \( A \) if

\[ x_1, x_2 \in A \quad \text{and} \quad x_1 < x_2 \quad \text{implies} \quad f(x_1) \geq f(x_2). \]

(iii) We say that \( f \) is strictly increasing/ decreasing if inequalities in (i) / (ii) are strict.

Geometrically,
Definitions:

Let $f : [a, b] \to \mathbb{R}$.

(i) We say $f$ has a local maximum at $c \in (a, b)$ if there exists $\delta > 0$ such that for $x \in A$, $c - \delta < x < c + \delta$ implies $f(x) \leq f(c)$.

(ii) We say $f$ has a local minimum at $c \in (a, b)$ if there exists $\delta > 0$ such that for $x \in A$, $c - \delta < x < c + \delta$ implies $f(x) \geq f(c)$.

(iii) We say $f$ has a local maximum at $a$, if there exists $\delta > 0$ such that for $x \in (a, b)$, $a < x < a + \delta$ implies $f(x) \leq f(a)$.

(iv) We say $f$ has a local minimum at $b$, if there exists $\delta > 0$ such that for $x \in (a, b)$, $b - \delta < x < b$ implies $f(x) \geq f(b)$. 
Examples:

(i) Let \( f(x) = x^2, -1 \leq x \leq 1 \). Then \( f \) has a local minimum at \( x = 0 \), since
\[
f(0) = 0 \leq x^2 \quad \forall x \in [-1, 1].
\]
Also, \( f \) has a local maximum at \( x = 1 \) and \( x = -1 \), because
\[
f(-1) = f(1) = 1 \geq f(x) \quad \forall x \in [-1, 1].
\]

(ii) The function \( f(x) = x^3, \quad a \leq x \leq b \) is always increasing, since
\[
\forall x, y \in [a, b], \quad x^3 < y^3 \quad \text{if} \quad x < y
\]
Let \( f(x) = \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 1. \end{cases} \)

Then \( f \) is increasing in \((0,1)\), \( f \) has a local maximum at \( x = 0 \) and a local minimum at \( x = 1 \).

**Lemma (Necessary condition for local extremum):**

If \( f : [a, b] \to \mathbb{R} \) is differentiable at \( c \in (a, b) \) and has a local maximum or a local minimum at \( c \), then \( f'(c) = 0 \).

**Proof:**

Suppose \( f \) has a local maximum at \( c \in (a, b) \). Using definition, there is a \( \delta > 0 \) such that \( f(x) \leq f(c) \) for every \( x \in (c - \delta, c + \delta) \subset (a, b) \).

Thus,

\[
\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.
\]
Hence, $f'(c) = 0$, the case of a local minimum at $c$ is similar

**Remark:**

(i) Above Lemma gives only a necessary condition for a function to have local maximum or minimum at a point. The conditions are not sufficient, i.e., the converse need not hold. For example, let $f(x) = x^3$. Then $f'(0) = 0$ but $f$ has no maximum/minimum at 0.

(ii) Lemma holds only for $c$ being an interior point. If $c$ is an end point, then $f$ can have a local max/min at $x = c$ without derivative being zero. For example, the function $f(x) = x, x \in [0, 1]$ has local maxima at $x = 1$ and local minima at $x = 0$ with $f'(0^+) = f'(1^-) = 1$.

(iii) $f$ Can have a local maximum/minimum at a point without being differentiable or even being continuous. For example, let

$$f(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Then $f$ has local maximum at $x = 0$, but $f$ is not even continuous at $x = 0$.

An important consequence of is the following:
Rolle's Theorem

If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, \( f' \) exists on \((a, b)\) and \( f(a) = f(b) \), then there exists at least one point \( c \in (a, b) \) such that \( f'(c) = 0 \).

Proof:

Let \( c_1, c_2 \in [a, b] \) be such that
\[
 f'(c_1) = \max\{ f(x) : x \in [a, b] \} \quad \text{and} \quad f'(c_2) = \min\{ f(x) : x \in [a, b] \}
\]

Note that such points \( c_1, c_2 \) exist as \( f \) is continuous on \([a, b]\). Either, \( c_1 \) or \( c_2 \) is an interior point of \([a, b]\). In which case
\[
 f''(c_1) = 0 \quad \text{or} \quad f''(c_2) = 0,
\]
by the preceding lemma. If not, then both \( c_1 \) and \( c_2 \) are end points of \([a, b]\). Now,
\[
 f(a) = f(b) \implies f(c_1) = f(c_2),
\]
and hence, \( f \) is constant on \([a, b]\). Thus, \( f'(c) = 0 \) for every \( c \in (a, b) \).
Examples:

(i) Let \( f(x) = x^2 - 2x \) on \([0, 2]\).

Then \( f \) is differentiable on \([0, 2]\) and \( f(0) = f(2) = 0 \). Thus, by Roll's theorem, there exists \( c \in (0, 2) \) such that \( f'(c) = 0 \).

In our case,
\[
f'(x) = 2x - 2 = 0 \implies x = 1.
\]

Thus for \( c = 1 \in (0, 2) \), \( f'(c) = 0 \).

(ii) Let \( f(x) = x^4 - 2x^2 \), \( x \in [-1, 1] \). Since \( f \) is differentiable on \([-1, 1]\) and \( f'(x) = -1 = f(-1) \), by Roll's theorem, there exists \( c \in (-1, 1) \) such that \( f'(c) = 0 \). In our case
\[
f'(x) = 4x^3 - 4x = 4x(x^2 - 1).
\]

Thus, \( f'(x) = 0 \) will hold for \( x = 0, \pm 1 \). However, \( c = 0 \in (-1, 1) \) only satisfies the required conclusion of Roll's theorem.

Remark:

In Rolle's Theorem, the continuity condition for the function \( f \) on the closed interval \([a, b]\) is essential, it cannot be weakened. For example, let

\[ f : [0, 1] \to \mathbb{R}, f(x) = x \text{ if } 0 \leq x < 1 \text{ and } f(1) = 0. \]

Then, \( f \) is continuous on \((0, 1)\),
\[ f(0) = f(1) \text{ but, } f'(x) = 1 \neq 0 \text{ for every } x \in (0, 1). \]
Examples:

(i) Let us see how Roll's Theorem is helpful in locating zeros of polynomials. Let

\[ f(x) = x^4 + 2x^3 - 2, \quad x \in [0,1]. \]

Note that, \( f \) is continuous with

\[ f(0) = -2 < 0 \quad \text{and} \quad f(1) = 1 > 0. \]

Thus, by the intermediate value property, \( f \) has at least one root in \((0,1)\). Suppose that \( f \) has two roots \( c_1, c_2 \) in \([0,1]\). Then by Roll's Theorem, \( f''(c) = 0 \) for some \( c \in (c_1, c_2) \).

But

\[ f''(x) = 4x^3 + 6x^2 > 0 \quad \text{for all} \quad x \in (0,1). \]

Hence, \( f \) can have at most one root in \([0,1]\) implying, \( f \) has a unique root in \((0,1)\).

(ii) Let

\[ f(x) = |x|, \quad x \in [-1,1]. \]

Then, \( f \) is continuous on \([-1,1]\). Even though, \( f(-1) = f(1) \),

\[ f'(x) \neq 0 \quad \text{for any} \quad x \in [-1,1]. \]

In fact,

\[ f'(x) = 1 \quad \text{or} \quad -1 \quad \text{for} \quad x \neq 0. \]

This does not contradict Rolle's theorem, since \( f'(0) \) does not exist.

(iii) Let \( f(x) = x, \quad x \in [0,1] \). Then \( f \) is continuous on \([0,1] \), \( f' \) exists but is nonzero on \((0,1)\). This does not contradict Rolle's theorem since \( f(0) \neq f(1) \).

We prove next an extension of the Rolle's theorem.
Lagrange's Mean Value Theorem (MVT):

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f'$ exists on $(a, b)$. Then there is at least one point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof:

The idea is to apply Rolle's Theorem to a suitable function $h : [a, b] \to \mathbb{R}$ such that $h(a) = h(b)$ and $h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$, $\forall x$.

From the figure, it is clear that such a $h(x)$ should be the difference between $f(x)$ and $L(x)$, the line joining $(a, f(a)$ and $(b, f(b))$. Thus, we consider $h(x)$ for $x \in [a, b]$.

Observe that $h$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $h(b) = 0$, $h(a) = 0$, i.e., $h(a) = h(b)$.

Hence, by Rolle's theorem, $h''(c) = 0$ for some $c \in (a, b)$, i.e.,

$$f''(c) = \frac{f(b) - f(a)}{b - a}$$

Physical Interpretation of MVT:
Let \( f: [a, b] \rightarrow \mathbb{R} \) denote the distance traveled by a body from time \( t = a \) to \( t = b \). Then, the average speed of a moving body between two points \( A \), at \( t = a \), and \( B \), at \( t = b \), is

\[
\text{Average speed} = \alpha := \frac{f(b) - f(a)}{b - a}.
\]

The mean value theorem says that there exists a time point \( t = c \) in between \( t = a \) and \( t = b \) when the speed of the body is actually \( \alpha \) km/sec.

**Theorem (Some Consequences of MVT):**

(i) Let \( f \) be differentiable on \((a, b)\). If \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f \) is constant on \((a, b)\).

(ii) Let \( f \) and \( g \) be differentiable on \((a, b)\). If \( g'(x) = f'(x) \) for all \( x \in (a, b) \), then there exists a real constant \( C \) such that \( g(x) = f(x) + C \forall x \in (a, b) \).

(iii) Let \( f \) be continuous on \([a, b]\) and differentiable on \((a, b)\).

If \( m, M \in \mathbb{R} \) are such that \( m \leq f'(x) \leq M \) for all \( x \in (a, b) \), then

\[
m(b - a) \leq f(b) - f(a) \leq M(b - a)
\]

**Example (Approximating square roots):**

Mean value theorem finds use in proving inequalities. For example, for \( n \in \mathbb{N} \), consider the function

\[
f(x) = \sqrt{x}, x \in [n, n+1].
\]

We have, by the mean value theorem,

\[
\sqrt{n+1} - \sqrt{n} = f(n+1) - f(n) = f'(c) = 1/(2\sqrt{c})
\]

for some \( c \in \mathbb{R} \) such that \( n < c < n + 1 \). Hence,

\[
1/(2\sqrt{n+1}) < \sqrt{n+1} - \sqrt{n} < 1/(2\sqrt{n})
\]

For \( n = 1 \), this gives \( \sqrt{2} < 1.5 \). Similarly, for \( n = 3 \) and \( n = 4 \), we get
We give yet another extension of Rolle's Theorem.

**Theorem (Cauchy's Mean Value Theorem):**

Let \( f, g \) defined on \([a, b]\) be continuous functions such that both are differentiable on \((a, b)\). Then, there exists \( c \in (a, b) \) such that

\[
[f(b) - f(a)]g'(c) = f'(c)[g(b) - g(a)].
\]

**Proof:**

If \( g(b) = g(a) \), we apply Rolle's Theorem to \( g \) to get a point \( c \in (a, b) \) such that \( g'(c) = 0 \). Then

\[
[f(b) - f(a)]g'(c) = 0 = f'(c)[g(b) - g(a)].
\]

In the case \( g(b) \neq g(a) \), define \( h: [a, b] \to \mathbb{R} \) by

\[
h(x) = f(x) - \alpha g(x),
\]

where \( \alpha \) is so chosen that \( h(b) = h(a) \), i.e.,

\[
\alpha = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

Now an application of Rolle's Theorem to \( h \) gives \( h'(c) = 0 \), for some \( c \in (a, b) \). Thus,

\[
0 = h'(c) = f'(c) - \alpha g'(c) = \left( \frac{f(b) - f(a)}{g(b) - g(a)} \right) g'(c)
\]

which gives the required equality.

**Practice Exercise**: Rolle's theorem and mean value theorems

(1) Show that the following functions satisfy conditions of the Rolle's theorem. Find a point \( c \), as given by the Rolle's theorem for which \( f'(c) = 0 \):

(i) \( f(x) = (x-1)(x-2)(x-3), \ x \in [1, 3] \).
(2) Verify that the hypothesis of the Mean Value theorem are satisfied for the given function on the
given interval. Also find all points \( c \) given by the theorem:

(i) \( f(x) = x^3 - x - 4, \quad x \in [-1, 2] \)

(ii) \( f(x) = 2x + \frac{1}{x}, \quad x \in [3, 4] \)

(iii) \( f(x) = x(x^2 - x - 2), \quad x \in [-1, 1] \)

(3) Let \( f(t) = At^2 + Bt + C \) be the distance traveled by a body for \( t \in [a, b] \). Show that the average
speed of the body is always attained at the mid point:

\[ t = \frac{a + b}{2} \]

(4) Let \( P \) and \( Q \) be two real numbers with \( P > 0 \). Show that the cubic \( x^3 + px + q \) has exactly one real
root.

(5) Show that the cubic \( x^3 - 6x + 3 \) has all roots real.

(6) Let \( f \) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) \) and \( f(b) \) are of different
signs and

\[ f'(x) \neq 0 \quad \text{for all} \quad x \in (a, b), \]

then there is a unique \( x_0 \in (a, b) \) such that \( f(x_0) = 0 \).

(7) Consider the cubic \( f(x) = x^3 + px + q \), where \( p \) and \( q \) are real numbers. If \( f(x) \) has three
distinct real roots,

then show that \( 4p^3 + 27q^2 < 0 \) by proving the following:

(i) \( p < 0 \).

(ii) \( f \) has maxima at \( -\sqrt{-\frac{p}{3}} \) and minima at \( \sqrt{-\frac{p}{3}} \).

\[ f\left(-\sqrt{-\frac{p}{3}}\right) f\left(\sqrt{-\frac{p}{3}}\right) < 0 \].
(8) Let \( n \in \mathbb{N} \) and \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is continuous on \([a, b]\) and \( f^{(n)} \) exists in \((a, b)\). If \( f \) vanishes at \( n+1 \) distinct points in \([a, b]\), then show that \( f^{(n)} \) vanishes at least once in \((a, b)\).

(9) Let \( f, g, h \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Show that there is some \( c \in (a, b) \) such that
\[
\begin{vmatrix} f(a) & f(b) & f'(c) \\ g(a) & g(b) & g'(c) \\ h(a) & h(b) & h'(c) \end{vmatrix} = 0.
\]
Deduce that if \( h(x) = 1 \) for all \( x \in [a, b] \), we obtain the conclusion of Cauchy's Mean Value Theorem, i.e.,
\[
[f(b) - f(a)] g'(c) = f'(c) [g(b) - g(a)].
\]
What does the result say if \( g(x) = x \) and \( h(x) = 1 \) for all \( x \in [a, b] \)?

(10) Use the Mean Value Theorem to prove \( |\sin a - \sin b| \leq |a-b| \) for all \( a, b \in \mathbb{R} \).

(11) Let \( f : [0, \pi/2] \to \mathbb{R} \) be continuous and satisfy \( f'(x) = \frac{1}{2} (1 + \cos x) \) for all \( x \in (0, \pi/2) \). If \( f(0) = 3 \), estimate \( f(\pi/2) \).

(12) Let \( f \) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = a \) and \( f(b) = b \), show that there exist distinct \( c_1, c_2 \in (a, b) \) such that \( f'(c_1), f'(c_2) = 2 \). Formulate and prove a similar result for \( \mathbb{R} \) points \( c_1, \ldots, c_n \in (a, b) \).

(13) Let \( a > 0 \) and \( f \) be continuous on \([-a, a]\). Suppose that \( f'(x) \) exists and \( f'(x) \leq 1 \) for all \( x \in (-a, a) \). If \( f(a) = a \) and \( f(-a) = -a \), show that \( f(0) = 0 \).

(14) In each case, find a function \( f \) which satisfies all the given conditions, or else show that no such function exists.

(i) \( f''(x) > 0 \) for all \( x \in \mathbb{R} \), \( f'(0) = 1 \), \( f'(1) = 1 \)

(ii) \( f''(x) > 0 \) for all \( x \in \mathbb{R} \), \( f'(0) = 1 \), \( f'(1) = 2 \)

(iii) \( f''(x) > 0 \) for all \( x \in \mathbb{R} \), \( f'(0) = 1 \), \( f(x) \leq 100 \) for all \( x > 0 \)

(iv) \( f''(x) > 0 \) for all \( x \in \mathbb{R} \), \( f'(0) = 1 \), \( f(x) \leq 1 \) for all \( x < 0 \).
(15) (Intermediate value Property for $f'$): Let $f$ be differentiable on $[a, b]$. Show that the function $f'$ has the Intermediate Value Property on $[a, b]$.

(Hint: If $f'(a) < r < f'(b)$, then the function $g$ defined by $g(x) = f(x) - rx, x \in [a, b]$, does not assume its minimum at or at $a$ at $b$.)