System of Equations: An Introduction

Many books on linear algebra will introduce matrices via systems of linear equations. We tried a different approach. We hope this way you will appreciate matrices as a powerful tool useful not only to solve linear systems of equations. Basically, the problem of finding some unknowns linked to each others via equations is called a system of equations. For example,

$$\begin{cases} 2x - 3y &= 1 \\ x + 3y &= -2 \end{cases}$$

and

$$\begin{cases} x^2 + y^2 &= 1 \\ x + 3y &= -2 \end{cases}$$

are systems of two equations with two unknowns (*x* and *y*), while

$$\begin{cases} 2x - 3y^2 &= -1 \\ x + y + z &= 1 \end{cases}$$

is a system of two equations with three unknowns (*x*, *y*, and *z*).

These systems of equations occur naturally in many real life problems. For example, consider a nutritious drink which consists of whole egg, milk, and orange juice. The food energy and protein for each of the ingredients are given by the table:

	Food Energy (Calories)	Protein (Grams)
1 egg	80	6
1 cup milk	160	9
1 cup orange juice	110	2

A natural question to ask is how much of each ingredient do we need to produce a drink of 540 calories and 25 grams of protein. In order to answer that, let *x* be the number of eggs, *y* the amount of milk (in cups), and *z* the amount of orange of juice (in cups). Then we need to have

$$\left\{\begin{array}{lcl} 80x + 160y + 110z & = & 540 \\ 6x + 9y + 2z & = & 25 \end{array}\right.$$

The task of **Solving** a system consists of finding the unknowns, here: *x*, *y* and *z*. A solution is a set of numbers once substituted for the unknowns will satisfy the equations of the system. For example, (2,1,2) and (0.325, 2.25, 1.4) are solutions to the system above.

The fundamental problem associated to any system is to find all the solutions. One way is to study the structure of its set of solutions which, in some cases, may help finding the solutions. Indeed, for example, in order to find the solutions to a linear system, it is enough to find just a few of them. This is possible because of the rich structure of the set of solutions.

Systems of Linear Equations: Gaussian Elimination

It is quite hard to solve non-linear systems of equations, while linear systems are quite easy to study. There are numerical techniques which help to approximate nonlinear systems with linear ones in the hope that the solutions of the linear systems are close enough to the solutions of the nonlinear systems. We will not discuss this here. Instead, we will focus our attention on linear systems.

For the sake of simplicity, we will restrict ourselves to three, at most four, unknowns. The reader interested in the case of more unknowns may easily extend the following ideas.

Definition. The equation

$$a\,x + b\,y + c\,z + d\,w = h$$

where a, b, c, d, and h are known numbers, while x, y, z, and w are unknown numbers, is called a **linear equation**. If h =0, the linear equation is said to be **homogeneous**. A **linear system** is a set of linear equations and a **homogeneous linear system** is a set of homogeneous linear equations.

For example,

$$\begin{cases} 2x - 3y & = & 1 \\ x + 3y & = & -2 \end{cases}$$

and

$$\begin{cases} x + y - z & = & 1 \\ x + 3y + 3z & = & -2 \end{cases}$$

are linear systems, while

$$\begin{cases} 2x - 3y^2 & = & -1 \\ x + y + z & = & 1 \end{cases}$$

is a nonlinear system (because of y^2). The system

$$\begin{cases} 2x & - & 3y & - & 3z & + & w & = & 0 \\ x & + & 3y & & & & & = & 0 \\ x & - & y & + & & & w & = & 0 \end{cases}$$

is an homogeneous linear system.

Matrix Representation of a Linear System

Matrices are helpful in rewriting a linear system in a very simple form. The algebraic properties of matrices may then be used to solve systems. First, consider the linear system

$$\begin{cases} ax+by+cz+dw &= e \\ fx+gy+hz+iw &= j \\ kx+ly+mz+nw &= p \\ qx+ry+sz+tw &= u \end{cases}.$$

Set the matrices

$$A = \begin{pmatrix} a & b & c & d \\ f & g & h & i \\ k & l & m & n \\ q & r & s & t \end{pmatrix},\ C = \begin{pmatrix} e \\ j \\ p \\ u \end{pmatrix},\ \text{and}\ X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Using matrix multiplications, we can rewrite the linear system above as the matrix equation

$$A \cdot X = C.$$

As you can see this is far nicer than the equations. But sometimes it is worth to solve the system directly without going through the matrix form. The matrix *A* is called the matrix coefficient of the linear system. The matrix *C* is called the **nonhomogeneous term**. When $C = \mathcal{O}$, the linear system is homogeneous. The matrix *X* is the unknown matrix. Its entries are the unknowns of the linear system. The **augmented matrix** associated with the system is the matrix [*A*|*C*], where

$$[A|C] == \left(\begin{array}{cccc|c} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & p \\ q & r & s & t & u \end{array}\right).$$

In general if the linear system has n equations with m unknowns, then the matrix coefficient will be a nxm matrix and the augmented matrix an nx(m+1) matrix. Now we turn our attention to the solutions of a system.

Definition. Two linear systems with n unknowns are said to be **equivalent** if and only if they have the same set of solutions.

This definition is important since the idea behind solving a system is to find an equivalent system which is easy to solve. You may wonder how we will come up with such system? Easy, we do that through **elementary operations**. Indeed, it is clear that if we interchange two equations, the new system is still equivalent to the old one. If we multiply an equation with a nonzero number, we obtain a new system still equivalent to old one. And finally replacing one equation with the sum of two equations, we again obtain an equivalent system. These operations are called **elementary operations** on systems. Let us see how it works in a particular case.

Example. Consider the linear system

$$\left\{\begin{array}{lcl} x+y+z & = & 0 \\ x-2y+2z & = & 4 \\ x+2y-z & = & 2 \end{array}\right.$$

The idea is to keep the first equation and work on the last two. In doing that, we will try to kill one of the unknowns and solve for the other two. For example, if we keep the first and second equation, and subtract the first one from the last one, we get the equivalent system

$$\left\{\begin{array}{rrcl} x+ & y+z & = & 0 \\ x- & 2y+2z & = & 4 \\ & y-2z & = & 2 \end{array}\right.$$

Next we keep the first and the last equation, and we subtract the first from the second. We get the equivalent system

$$\left\{\begin{array}{rrcl} x+ & y+z & = & 0 \\ - & 3y+z & = & 4 \\ & y-2z & = & 2 \end{array}\right.$$

Now we focus on the second and the third equation. We repeat the same procedure. Try to kill one of the two unknowns (*y* or *z*). Indeed, we keep the first and second equation, and we add the second to the third after multiplying it by 3. We get

$$\begin{cases} x+ & y+ & z & = & 0 \\ - & 3y+ & z & = & 4 \\ & - & 5z & = & 10 \end{cases}$$

This obviously implies *z* = -2. From the second equation, we get *y* = -2, and finally from the first equation we get *x* = 4. Therefore the linear system has one solution

$$x = 4,\ y = -2,\ z = -2.$$

Going from the last equation to the first while solving for the unknowns is called **backsolving**.

Keep in mind that linear systems for which the matrix coefficient is upper-triangular are easy to solve. This is particularly true, if the matrix is in echelon form. So the trick is to perform elementary operations to transform the initial linear system into another one for which the coefficient matrix is in echelon form.
Using our knowledge about matrices, is there anyway we can rewrite what we did above in matrix form which will make our notation (or representation) easier? Indeed, consider the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right).$$

Let us perform some elementary row operations on this matrix. Indeed, if we keep the first and second row, and subtract the first one from the last one we get

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right).$$

Next we keep the first and the last rows, and we subtract the first from the second. We get

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right).$$

Then we keep the first and second row, and we add the second to the third after multiplying it by 3 to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & -5 & 10 \end{array}\right).$$

This is a triangular matrix which is not in echelon form. The linear system for which this matrix is an augmented one is

$$\begin{cases} x+ & y+ & z & = & 0 \\ - & 3y+ & z & = & 4 \\ & - & 5z & = & 10 \end{cases}$$

As you can see we obtained the same system as before. In fact, we followed the same elementary operations performed above. In every step the new matrix was exactly the augmented matrix associated to the new system. This shows that instead of writing the systems over and over again, it is easy to play around with the elementary row operations and once we obtain a triangular matrix, write the associated linear system and then solve it. This is known as **Gaussian Elimination**. Let us summarize the procedure:

Gaussian Elimination. Consider a linear system.

1. Construct the augmented matrix for the system;

2. Use elementary row operations to transform the augmented matrix into a triangular one;

3. Write down the new linear system for which the triangular matrix is the associated augmented matrix;

4. Solve the new system. You may need to assign some parametric values to some unknowns, and then apply the method of back substitution to solve the new system.

Example. Solve the following system via Gaussian elimination

$$\begin{cases} 2x- & 3y-z+ & 2w & +3v & = & 4 \\ 4x- & 4y-z+ & 4w & +11v & = & 4 \\ 2x- & 5y-2z+ & 2w & -v & = & 9 \\ & 2y+z & & +4v & = & -5 \end{cases}$$

The augmented matrix is

$$\left(\begin{array}{rrrrr|r} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{array}\right).$$

We use elementary row operations to transform this matrix into a triangular one. We keep the first row and use it to produce all zeros elsewhere in the first column. We have

$$\left(\begin{array}{rrrrr|r} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & -2 & -1 & 0 & -4 & 5 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{array}\right).$$

Next we keep the first and second row and try to have zeros in the second column. We get

$$\left(\begin{array}{rrrrr|r} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right).$$

Next we keep the first three rows. We add the last one to the third to get

$$\left(\begin{array}{rrrrr|r} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This is a triangular matrix. Its associated system is

$$\left\{\begin{array}{rcrcrcrcrcr} 2x- & 3y & - & z & + & 2w & + & 3v & = & 4 \\ & 2y & + & z & & & + & 5v & = & -4 \\ & & & & & & & v & = & 1 \end{array}\right.$$

Clearly we have $v = 1$. Set $z=s$ and $w=t$, then we have

$$y = -2 - \frac{1}{2}z - \frac{5}{2}v = -\frac{9}{2} - \frac{1}{2}s.$$

The first equation implies

$$x = 2 + \frac{3}{2}y + \frac{1}{2}z - w - \frac{3}{2}v.$$

Using algebraic manipulations, we get

$$x = -\frac{25}{4} - \frac{1}{4}s - t.$$

Putting all the stuff together, we have

$$\begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} -\frac{25}{4} - \frac{1}{4}s - t \\ -\frac{9}{2} - \frac{1}{2}s \\ s \\ t \\ 1 \end{pmatrix}.$$

Example. Use Gaussian elimination to solve the linear system

$$\left\{ \begin{array}{rcr} x - y & = & 4 \\ 2x - 2y & = & -4 \end{array} \right.$$

The associated augmented matrix is

$$\left(\begin{array}{rr|r} 1 & -1 & 4 \\ 2 & -2 & -4 \end{array} \right).$$

We keep the first row and subtract the first row multiplied by 2 from the second row. We get

$$\left(\begin{array}{rr|r} 1 & -1 & 4 \\ 0 & 0 & -12 \end{array} \right).$$

This is a triangular matrix. The associated system is

$$\begin{cases} x - y = 4 \\ 0 = -12 \end{cases}$$

Clearly the second equation implies that this system has no solution. Therefore this linear system has no solution.

Definition. A linear system is called **inconsistent or overdetermined** if it does not have a solution. In other words, the set of solutions is empty. Otherwise the linear system is called **consistent**.

Following the example above, we see that if we perform elementary row operations on the augmented matrix of the system and get a matrix with one of its rows equal to $(0, 0, \cdots, 0, c)$, where $c \neq 0$, then the system is inconsistent.

SYSTEMS OF EQUATIONS in TWO VARIABLES

A system of equations is a collection of two or more equations with the same set of unknowns. In solving a system of equations, we try to find values for each of the unknowns that will satisfy every equation in the system.

The equations in the system can be linear or non-linear. This tutorial reviews systems of linear equations.

A problem can be expressed in narrative form or the problem can be expressed in algebraic form.

Let's start with an example stated in narrative form. We'll convert it to an equivalent equation in algebraic form, and then we will solve it.

Example 1:

A total of $12,000 is invested in two funds paying 9% and 11% simple interest. If the yearly interest is $1,180, how much of the $12,000 is invested at each rate?

Before you work this problem, you must know the definition of simple interest. Simple interest can be calculated by multiplying the amount invested at the interest rate.

Solution:

We have two unknowns: the amount of money invested at 9% and the amount of money invested at 11%. Our objective is to find these two numbers.

Sentence (1) "A total of $12,000 is invested in two funds paying 9% and 11% simple interest." can be restated as (The amount of money invested at 9%) $+$ (The amount of money invested at 11%) $=$ $12,000.

Sentence (2) "If the yearly interest is $1,180, how much of the $12,000 is invested at each rate?" can be restated as (The amount of money invested at 9%) $\times$ 9% + (The amount of money invested at 11% $\times$ 11%) $=$ total interest of $1,180.

It is going to get tiresome writing the two phrases (The amount of money invested at 9%) and (The amount of money invested at 11%) over and over again. So let's write them in shortcut form. Call the phrase (The amount of money invested at 9%) by the symbol x and call the phrase (The amount of money invested at 11%) by the symbol y.

Let's rewrite sentences (1) and (2) in shortcut form.

$$(1) \quad : \quad x + y = \$12,000.$$

$$(2) \quad : \quad 0.09x + 0.11y = \$1180$$

We have converted a narrative statement of the problem to an equivalent algebraic statement of the problem. Let's solve this system of equations.

A system of linear equations can be solved four different ways:

Substitution,

Elimination,

Matrices,

Graphing.

The Method of Substitution:

The method of substitution involves five steps:

Step 1: Solve for y in equation (1).

$$\begin{aligned} x + y &= \$12,000. \\ y &= \$12,000 - x \end{aligned}$$

Step 2: Substitute this value for y in equation (2). This will change equation (2) to an equation with just one variable, x.

$$\begin{aligned} 0.09x + 0.11y &= \$1180 \\ 0.09x + 0.11\left(\$12,000 - x\right) &= \$1180 \end{aligned}$$

Step 3: Solve for x in the translated equation (2).

$$\begin{aligned} 0.09x + 0.11\,(\$12,000 - x) &= \$1180 \\ 0.09x + \$1,320 - 0.11x &= \$1,180 \\ -0.02x &= -\$140 \\ x &= \$7,000 \end{aligned}$$

Step 4: Substitute this value of x in the y equation you obtained in Step 1.

$$(1) \quad : \quad x + y = \$12,000.$$

$$\begin{aligned} \$7,000 + y &= \$12,000 \\ y &= \$5,000 \end{aligned}$$

Step 5: Check your answers by substituting the values of x and y in each of the original equations. If, after the substitution, the left side of the equation equals the right side of the equation, you know that your answers are correct.

$$(1) \quad : \quad x + y = \$12,000.$$

$$\$7,000 + \$5,000 \quad = \quad \$12,000$$

$$(2) \quad : \quad 0.09x + 0.11y = \$1180$$

$$0.09\,(\$7,000) + 0.11\,(\$5,000) \quad = \quad \$1180$$

The Method of Elimination:

The process of elimination involves five steps:

In a two-variable problem rewrite the equations so that when the equations are added, one of the variables is eliminated, and then solve for the remaining variable.

Step 1: Change equation (1) by multiplying equation (1) by -0.09 to obtain a new and equivalent equation (1).

$$(1) \quad : \quad x + y = \$12,000.$$

$$(New\ 1) \quad : \quad -0.09x - 0.09y = -\$,1080$$

Step 2: Add new equation (1) to equation (2) to obtain equation (3).

$$\begin{aligned} (New\ 1) &\quad : \quad -0.09x - 0.09y = -\$,1080 \\ (2) &\quad : \quad 0.09x + 0.11y = \$1180 \\ (3) &\quad : \quad 0.02y = 100 \\ y &\quad = \quad \$5,000 \end{aligned}$$

Step 3: Substitute $y = \$5,000$ in equation (1) and solve for x.

$$\begin{aligned} x + y &= \$12,000 \\ \\ x + \$5,000 &= \$12,000 \\ x &= \$7,000 \end{aligned}$$

Step 4: Check your answers in equation (2). Does

$$0.09x + 0.11y = \$1,180$$

$$0.09\,(\$7,000) + 0.11\$5,000 = \$1,180$$

The Method of Matrices:

This method is essentially a shortcut for the method of elimination.

Rewrite equations (1) and (2) without the variables and operators. The left column contains the coefficients of the x's, the middle column contains the coefficients of the y's, and the right column contains the constants.

$$\begin{array}{c} (1) \\ (2) \end{array} \left[\begin{array}{cc|c} 1 & 1 & 12000 \\ 0.09 & 0.11 & 1180 \end{array}\right]$$

The objective is to reorganize the original matrix into one that looks like

$$\begin{array}{c} (1) \\ (2) \end{array} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

where a and b are the solutions to the system.

Step 1. Manipulate the matrix so that the number in cell 11 (row 1-col 1) is 1. In this case, we don't have to do anything. The number 1 is already in the cell.

Step 2: Manipulate the matrix so that the number in cell 21 is 0. To do this we rewrite the matrix by keeping row 1 and creating a new row 2 by adding -0.09 x row 1 to row 2.

$$-0.09\,[Row\ 1] + [row\ 2] \quad = \quad [New\ Row\ 2]$$

$$\begin{array}{c} (1) \\ \\ (2) \end{array} \begin{bmatrix} 1 & 1 & | & 12000 \\ & & | & \\ 0 & 0.02 & | & 100 \end{bmatrix}$$

Step 3: Manipulate the matrix so that the cell 22 is 1. Do this by multiplying row 2 by 50.

$$\begin{array}{c} (1) \\ \\ (2) \end{array} \begin{bmatrix} 1 & 1 & | & 12000 \\ & & | & \\ 0 & 1 & | & 5000 \end{bmatrix}$$

Step 4: Manipulate the matrix so that cell 12 is 0. Do this by adding

$$-1[Row\ 2] + [Row\ 1] \quad = \quad [New\ Row\ 1]$$

$$\begin{array}{c} (1) \\ (2) \end{array} \left[\begin{array}{cc|c} 1 & 0 & 7000 \\ 0 & 1 & 5000 \end{array}\right]$$

You can read the answers off the matrix as x = $7,000 and y = $5,000.

The method of Graphing:

In this method solve for y in each equation and graph both. The point of intersection is the solution

SYSTEMS OF EQUATIONS in THREE VARIABLES

It is often desirable or even necessary to use more than one variable to model a situation in a field such as business, science, psychology, engineering, education, and sociology, to name a few. When this is the case, we write and solve a system of equations in order to answer questions about the situation.

If a system of linear equations has at least one solution, it is **consistent**. If the system has no solutions, it is **inconsistent**. If the system has an infinity number of solutions, it is **dependent**. Otherwise it is **independent**.

A linear equation in three variables is an equation equivalent to the equation
$Ax + By + Cz + D = 0$

where A, B, C, and D are real numbers and A, B, C, and D are not all 0.

Example 1:
John inherited $25,000 and invested part of it in a money market account, part in municipal bonds, and part in a mutual fund. After one year, he received a total of $1,620 in simple interest from the three investments. The money market paid 6% annually, the bonds paid 7% annually, and the mutually fund paid 8% annually. There was $6,000 more invested in the bonds than the mutual funds. Find the amount John invested in each category.

There are three unknowns:
1 : The amount of money invested in the money market account.
2 : The amount of money invested in municipal bonds.
3 : The amount of money invested in a mutual fund.

Let's rewrite the paragraph that asks the question we are to answer.

[The amount of money invested in the money market account + [The amount of money invested in municipal bonds] + [The amount of money invested in a mutual fund] $= \$25,000.$

The 6% interest on [The amount of money invested in the money market account]+ the 7% interest on [The amount of money invested in municipal bonds] + the 8% interest on [The amount of money invested in a mutual fund] $= \$1,620$

[The amount of money invested in municipal bonds] - [The amount of money invested in a mutual fund] = $\$6,000$.

It is going to get boring if we keep repeating the phrases
1 : The amount of money invested in the money market account.
2 : The amount of money invested in municipal bonds.
3 : The amount of money invested in a mutual fund.

Let's create a shortcut by letting symbols represent these phrases. Let

x = The amount of money invested in the money market account.
y = The amount of money invested in municipal bonds.
z = The amount of money invested in a mutual fund.

in the three sentences, and then rewrite them.

The sentence [The amount of money invested in the money market account] $+$ [The amount of money invested in municipal bonds] $+$ [The amount of money invested in a mutual fund] $= \$25,000$ can now be written as
$x + y + z = \$25,000$

The sentence The 6% interest on [The amount of money invested in the money market account] $+$ the 7% interest on [The amount of money invested in municipal bonds] $+$ the 8% interest on [The amount of money invested in a mutual fund] $= \$1,620$ can now be written as
$0.06x + 0.07y + 0.08z = \$1,620$

The sentence [The amount of money invested in municipal bonds] $-$ [The amount of money invested in a mutual fund] $=$ $\$6,000$ can now be written as
$y - z = \$6,000$

We have converted the problem from one described by words to one that is described by three equations.

$$\begin{array}{rcrcrcrl} x & + & y & + & z & = & \$\,25,000 & \quad (1) \\ 0.06x & + & 0.07y & + & 0.08z & = & \$\,1,620 & \quad (2) \\ & & y & - & z & = & \$\,6,000 & \quad (3) \end{array}$$

We are going to show you how to solve this system of equations three different ways:
1) Substitution,
2) Elimination,
3) Matrices.

SUBSTITUTION:
The process of substitution involves several steps:
Step 1: Solve for one of the variables in one of the equations. It makes no difference which equation and which variable you choose. Let's solve for y in equation (3) because the equation only has two variables.
$y - z = \$6,000$

$y = \$6,000 + z$

Step 2: Substitute this value for y in equations (1) and (2). This will change equations (1) and (2) to equations in the two variables x and z. Call the changed equations (4) and (5).

$$
\begin{aligned}
x + y + z &= \$25,000 \\
x + (\$6,000 + z) + z &= \$25,000 \\
x + 2z &= \$19,000 \\
0.06x + 0.07y + 0.08z &= \$1,620 \\
0.06x + 0.07\,(\$6,000 + z) + 0.08z &= \$1,620 \\
0.06x + 0.15z &= \$1,200
\end{aligned}
$$

or

$$
\begin{array}{rcrcrl}
x & + & 2z & = & \$\,19,000 & \qquad (4) \\
0.06x & + & 0.15z & = & \$\,1,200 & \qquad (5)
\end{array}
$$

Step 3: Solve for x in equation (4).

$$
\begin{aligned}
x + 2z &= \$19,000 \\
x &= \$19,000 - 2z
\end{aligned}
$$

Step 4: Substitute this value of x in equation (5). This will give you an equation in one variable.

$$
\begin{aligned}
0.06x + 0.15z &= \$1,200 \\
0.06\,(\$19,000 - 2z) + 0.15z &= \$1,200 \\
0.03z &= 60
\end{aligned}
$$

Step 5: Solve for z.

$$0.03z = 60$$
$$z = \$2,000$$

Step 6: Substitute this value of z in equation (4) and solve for x.

$$x + 2z = \$19,000$$
$$x + 2\,(\$2,000) = \$19,000$$
$$x = \$15,000$$

Step 7: Substitute $\$15,000$ for x and $\$2,000$ for z in equation (1) and solve for y.

$$\$15,000 + y + \$2,000 = \$25,000$$
$$y = \$8,000$$

The solutions: $\$15,000$ is invested in the monkey market account, $\$8,000$ is invested in the municipal bonds, and $\$2,000$ is invested in mutual funds.

Step 8: Check the solutions:

$\$15,000 + \$8,000 + \$2,000 = \$25,000 \rightarrow$ Yes

$0.06\,(\$15,000) + 0.07\,(\$8,000) + 0.08\,(\$2,000) = \$1,620 \rightarrow$ Yes

$\$8,000 - \$2,000 = \$6,000 \rightarrow$ Yes

ELIMINATION:

The process of elimination involves several steps: First you reduce three equations to two equations with two variables, and then to one equation with one variable.

Step 1: Decide which variable you will eliminate. It makes no difference which one you choose. Let us eliminate x first because x is missing from equation (3).

$$\begin{aligned}(1)\ x + y + z &= \$25,000\\ (2)\ 0.06x + 0.07y + 0.08z &= \$1,620\\ (3)\ y - z &= \$6,000\end{aligned}$$

Step 2: Multiply both sides of equation (1) by -0.06 and then add the transformed equation (1) to equation (2) to form equation (4).

(1) : $-0.06x - 0.06y - 0.06z = -\$1,500$

(2) : $0.06x + 0.07y + 0.08z = \$1,620$

(4) : $0.01y + 0.02z = \$120$

Step 3: We now have two equations with two variables.

(3) : $y - z = \$6,000$

(4) : $0.01y + 0.02z = \$120$

Step 4: Multiply both sides of equation (3) by 0.02 and add to equation (4) to create equation (5) with just one variable.

$$(3): \quad 0.02y - 0.02z = \$120$$

$$(4): \quad 0.01y + 0.02z = \$120$$

$$(5): \quad 0.03y = \$240$$

Step 5: Solve for y in equation (5).

$0.03y = \$240$

$y = \$8,000$

Step 6: Substitute $\$8,000$ for y in equation (3) and solve for z.

$y - z = \$6,000$

$\$8,000 - z = \$6,000$

$z = \$2,000$

Step 7: Substitute $\$8,000$ for y and $\$2,000$ for z in equation (1) and solve for x.

$$\begin{aligned} x + y + z &= \$25,000 \\ x + \$8,000 + \$2,000 &= \$25,000 \\ x &= \$15,000 \end{aligned}$$

Check your answers as before.

MATRICES:

The process of using matrices is essentially a shortcut of the process of elimination. Each row of the matrix represents an equation and each column represents coefficients of one of the variables.

Step 1: Create a three-row by four-column matrix using coefficients and the constant of each equation.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \$25,000 \\ 0.06 & 0.07 & 0.08 & \$1,620 \\ 0 & 1 & -1 & \$6,000 \end{array}\right]$$

The vertical lines in the matrix stands for the equal signs between both sides of each equation. The first column contains the coefficients of x, the second column contains the coefficients of y, the third column contains the coefficients of z, and the last column contains the constants.

We want to convert the original matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \$25,000 \\ 0.06 & 0.07 & 0.08 & \$1,620 \\ 0 & 1 & -1 & \$6,000 \end{array}\right]$$

to the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

Because then you can read the matrix as $x = a$, $y = b$, and $z = c$.

Step 2: We work with column 1 first. The number 1 is already in cell 11(Row1-Col 1). Add -0.06 times Row 1 to Row 2 to form a new Row 2.

$-0.06\,[Row\ 1] + [Row\ 2] = [New\ Row\ 2]$

$$\begin{bmatrix} 1 & 1 & 1 & | & \$25,000 \\ 0 & 0.01 & 0.02 & | & \$120 \\ 0 & 1 & -1 & | & \$6,000 \end{bmatrix}$$

Step 3: We will now work with column 1. We want 1 in Cell 22, and we achieve this by multiply Row 2 by 100.

$100\,[Row\ 2] = [New\ Row\ 2]$

$$\begin{bmatrix} 1 & 1 & 1 & | & \$25,000 \\ 0 & 1 & 2 & | & \$12,000 \\ 0 & 1 & -1 & | & \$6,000 \end{bmatrix}$$

Step 4: Let's now manipulate the matrix so that there are zeros in Cell 12 and Cell 32. We do this by adding -1 times Row 2 to Row 1 and Row 3 for a new Row 1 and a new Row 3.

$$-[Row\ 2] + [Row\ 1] = [New\ Row\ 1]$$
$$-[Row\ 2] + [Row\ 3] = [New\ Row\ 3]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & \$13,000 \\ 0 & 1 & 2 & \$12,000 \\ 0 & 0 & -1 & -\$2,000 \end{array}\right]$$

Step 5: Let's now manipulate the matrix so that there is a 1 in Cell 33. We do this by multiplying Row 3 by -1.

$$-1\,[Row\ 3] = [New\ Row\ 3]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & \$13,000 \\ 0 & 1 & 2 & \$12,000 \\ 0 & 0 & 1 & \$2,000 \end{array}\right]$$

Step 6: Let's now manipulate the matrix so that there are zeros in Cell 13 and Cell 23. We do this by adding Row 3 to Row 1 for a new Row 1 and adding -2 times Row 3 to Row 2 for a new Row 3.

$$\begin{aligned} 1\,[Row\ 3] + [Row\ 1] &= [New\ Row\ 1] \\ -2\,[Row\ 3] + [Row\ 2] &= [New\ Row\ 2] \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \$15,000 \\ 0 & 1 & 0 & \$8,000 \\ 0 & 0 & 1 & \$2,000 \end{array}\right]$$

You can now read the answers off the matrix: $x = \$15,000$, $y = \$8,000$, and $z = \$2,000$. check your answers by the method described above.

Application of Determinant to Systems: Cramer's Rule

We have seen that determinant may be useful in finding the inverse of a nonsingular matrix. We can use these findings in solving linear systems for which the matrix coefficient is nonsingular (or invertible).

Consider the linear system (in matrix form)

$$A\,X = B$$

where A is the matrix coefficient, B the nonhomogeneous term, and X the unknown column-matrix. We have:

Theorem. The linear system $AX = B$ has a unique solution if and only if A is invertible. In this case, the solution is given by the so-called **Cramer's formulas**:

$$x_i = \frac{\det(A_i)}{detA}\ , \quad \text{for } i = 1, \cdots, n$$

where x_i are the unknowns of the system or the entries of X, and the matrix A_i is obtained from A by replacing the i^{th} column by the column B. In other words, we have

$$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{\det(A)}$$

where the b_i are the entries of B.

In particular, if the linear system $AX = B$ is homogeneous, meaning $B = \mathcal{O}$, then if A is invertible, the only solution is the trivial one, that is $X = \mathcal{O}$. So if we are looking for a nonzero solution to the system, the matrix coefficient A must be singular or noninvertible. We also know that this will happen if and only if $\det(A) = 0$. This is an important result.

Example. Solve the linear system

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Answer. First note that

$$\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = 9$$

which implies that the matrix coefficient is invertible. So we may use the Cramer's formulas. We have

$$x = \frac{1}{9} \begin{vmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}, \; y = \frac{1}{9} \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix}, \; \text{and } z = \frac{1}{9} \begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}.$$

We leave the details to the reader to find

$$x = \frac{-6}{9} = -\frac{2}{3}, \; y = \frac{3}{9} = \frac{1}{3}, \; \text{and } z = 0.$$

Note that it is easy to see that $z=0$. Indeed, the determinant which gives z has two identical rows (the first and the last). We do encourage you to check that the values found for x, y, and z are indeed the solution to the given system.

Remark. Remember that Cramer's formulas are only valid for linear systems with an invertible matrix coefficient.