Modified Regula Falsi method:

In this method an improvement over Regula Falsi method is obtained by replacing the secant by straight lines of even-smaller slope until \( w \) falls to the otherside of the zero of \( f(x) \). The various steps in the method are given in the algorithm below:

**Algorithm:**
Given a function \( f(x) \) continuous on an interval \([a, b]\) satisfying the criteria \( f(a)f(b) < 0 \), carry out the following steps to find the root of \( \xi \) of \( f(x) \) in \([a, b]\):

1. Set \( a_0 = a \); \( b_0 = b \); \( F = f(a_0) \); \( G = f(b_0) \); \( w_0 = a_0 \)
2. For \( n=0,1,2,\ldots \), until convergence criteria is satisfied, do:
   a. compute \( w_{n+1} = [Ga_n - Fb_n]/(G - F) \)
   b. If \( (f(w_n)f(w_{n+1}) > 0) \) then
      Set \( a_{n+1} = a_n \); \( b_{n+1} = w_{n+1} \); \( G = f(w_{n+1}) \)
      Also if \( (f(w_n)f(w_{n+1}) > 0) \) Set \( F = F/2 \)
   Otherwise
      Set \( a_{n+1} = w_{n+1} \), \( F = f(w_{n+1}) \) \( b_{n+1} = b_n \)
      Also if \( (f(w_n)f(w_{n+1}) > 0) \) Set \( G = G/2 \)

**Example:**
**Solve** \( 2x^3 - 2.5x - 5 = 0 \) for the root in the interval \([1,2]\) by Modified Regula Falsi method.

**Solution:** Since \( f(1) f(2) = -33 < 0 \), we go ahead with finding the root of given function \( f(x) \) in \([1,2]\). Setting \( a_0 = 1 \), \( b_0 = 2 \) and following the above algorithm.

Results are provided in the table below:
**Modified Regula Falsi Method**

<table>
<thead>
<tr>
<th>Iteration no.</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$\omega_n$</th>
<th>$f(\omega_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000000000</td>
<td>2.0000000000</td>
<td>1.4782608747</td>
<td>-2.2348976135</td>
</tr>
<tr>
<td>1</td>
<td>1.4782608747</td>
<td>2.0000000000</td>
<td>1.7010031939</td>
<td>0.5908976793</td>
</tr>
<tr>
<td>2</td>
<td>1.4782608747</td>
<td>1.7010031939</td>
<td>1.6544258595</td>
<td>-0.0793241411</td>
</tr>
<tr>
<td>3</td>
<td>1.6544258595</td>
<td>1.7010031939</td>
<td>1.6599385738</td>
<td>-0.0022699926</td>
</tr>
<tr>
<td>4</td>
<td>1.6599385738</td>
<td>1.7010031939</td>
<td>1.6602516174</td>
<td>0.0021237291</td>
</tr>
<tr>
<td>5</td>
<td>1.6599385738</td>
<td>1.6602516174</td>
<td>1.6601003408</td>
<td>0.0000002435</td>
</tr>
</tbody>
</table>

The geometric view of the example is provided in the figure below:

**Example:** Solve $5\sin x^2 - 8\cos^5 x = 0$ for the root in the interval $[0.5, 1.5]$ by Modified Regula Falsi Method.
<table>
<thead>
<tr>
<th>Iteration</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$\omega_n$</th>
<th>$f(\omega_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5000000000</td>
<td>1.5000000000</td>
<td>0.8773435354</td>
<td>2.1035263538</td>
</tr>
<tr>
<td>1</td>
<td>0.5000000000</td>
<td>0.8773435354</td>
<td>0.7222673893</td>
<td>0.2828366458</td>
</tr>
<tr>
<td>2</td>
<td>0.5000000000</td>
<td>0.7222673893</td>
<td>0.6871531010</td>
<td>-0.1967970580</td>
</tr>
<tr>
<td>3</td>
<td>0.6871531010</td>
<td>0.7222673893</td>
<td>0.7015607357</td>
<td>0.0026464546</td>
</tr>
<tr>
<td>4</td>
<td>0.6871531010</td>
<td>0.7015607357</td>
<td>0.7013695836</td>
<td>0.0000239155</td>
</tr>
<tr>
<td>5</td>
<td>0.6871531010</td>
<td>0.7013695836</td>
<td>0.7013661265</td>
<td>-0.0000235377</td>
</tr>
<tr>
<td>6</td>
<td>0.7013661265</td>
<td>0.7013695836</td>
<td>0.7013678551</td>
<td>-0.0000003363</td>
</tr>
</tbody>
</table>

**Secant Method**

Like the Regula Falsi method and the Bisection method this method also requires two initial estimates $x_{-1}$, $x_0$ of the root of $f(x)=0$ but unlike those earlier methods it gives up the demand of bracketing the root. Like in the Regula Falsi method, this method too retains the use of secants throughout while tracking the root of $f(x)=0$. The secant joining the points $(x_{-1}, f(x_{-1})), (x_0, f(x_0))$ is given by

$$y = \frac{f(x_0) - f(x_{-1})}{(x_0 - x_{-1})} x + \frac{f(x_{-1})x_0 - f(x_0)x_{-1}}{(x_0 - x_{-1})}$$

Say it intersects with x-axis at $x = x_1$, then

$$x_1 = \frac{f(x_0)x_{-1} - f(x_{-1})x_0}{f(x_0) - f(x_{-1})}$$

If $|f(x_1)| > \epsilon = 10^{-6}$ (say) then replace $(x_{-1}, f(x_{-1})), (x_0, f(x_0))$ with $(x_0, f(x_0)), (x_1, f(x_1))$ and repeat the process to get $x_1$ and so on. The method is algorithmically described below:

**Algorithm:**

Given a $f(x)$, two initial points $a, b$ and $\epsilon$ the required level of accuracy carry out the following steps to find the root $\xi$ of $f(x)=0$.

1. Set $x_{-1} = a, \quad x_0 = b$
(2) For \( n=0,1,2 \ldots \) until convergence criteria is satisfied, do:

Compute

\[
x_{n+1} = \frac{[f(x_n)x_{n-1} - f(x_{n-1})x_n]}{[f(x_n) - f(x_{n-1})]}
\]

Example:

Solve \( 2x^3 - 2.5x - 5 = 0 \) for the root with \( a = 1, b = 2 \) by secant method to an accuracy of \( 10^{-6} \).

Solution:

Set \( x_{-1} = a = 1 \) ; \( x_0 = b = 2 \)

\[
f(x_{-1}) = f(1) = -5.5
\]

\[
f(x_0) = f(2) = 6.0
\]

\[
x_1 = \frac{f(x_0)x_{-1} - f(x_{-1})x_0}{f(x_0) - f(x_{-1})}
\]

\[
= \frac{f(2).1 - f(1).2}{f(2) - f(1)}
\]

\[
= \frac{6.1 - (-5.5).2}{6 - (-5.5)}
\]

\[
= \frac{1.4782608747}{11.5} > 10^{-6}
\]

\[|f(x_1)| = |-2.2348976135| > 10^{-6}\]

\[\therefore\] Repeat the process with \( (x_0, f(x_0)), \ (x_1, f(x_1)) \) and so on till you get a \( \xi = x_i \)

s.t. \( |f(\xi)| < \epsilon = 10^{-6} \). These results are tabulated below:

**Secant Method**

<table>
<thead>
<tr>
<th>Iteration no.</th>
<th>( x_{n-1} )</th>
<th>( x_n )</th>
<th>( x_{n+1} )</th>
<th>( f(x_{n+1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000000000</td>
<td>2.00000000000</td>
<td>1.4782608747</td>
<td>-2.2348976135</td>
</tr>
<tr>
<td>1</td>
<td>2.00000000000</td>
<td>1.4782608747</td>
<td>1.6198574305</td>
<td>-0.5488323569</td>
</tr>
</tbody>
</table>
Geometrical visualization of the root tracking procedure by Secant method for the above example.

**Exercise:** Find the solutions accurate to within $10^{-4}$ for the following problems using Secant's Method.

(1) \[ x - x \cos x = 0, \ [0, \pi/2] \]

(2) \[ x - 0.8 - 0.2 \sin x = 0, \ [0, \pi/2] \]