Electrostatics

Quantization of Charge

Objectives
In this lecture you will learn the following

- Quantization Of Charge and its measurement
- Coulomb's Law of force between electric charge
- Superposition principle and concept of electric field
- Calculation of electric field for simple charge distributions and drawing of lines of force

QUANTIZATION OF CHARGE

Building blocks of matter are atoms, which consist of nucleus and electrons. Nucleus has positively charged protons and neutrons which are charge neutral. Negatively charged electrons move round the nucleus. It has been observed that the electric charge of all particles are integral multiple of an elemental value of charge. Denoting the magnitude of the charge of an electron by \( | e | \), the charge of all particles are

\[ \pm | e |, \pm 2 | e |, \pm 3 | e |, \ldots \]

This is known as **charge quantization**. Neutral particles, like neutron and photon have zero charge.

Physicists have revised their earlier thinking that particles like neutrons and protons are fundamental particles.

They are now regarded as belonging to a group of particles called **Hadrons**, which are built up of fundamental constituents called **quarks**, which have fractional charge of magnitude one third or two third that of an electronic charge. Electrons, on the other hand are considered elementary particles, belonging to a group called **Leptons**.
Millikan's Oil Drop Experiment

The quantization of charge was experimentally established by Robert Millikan in 1909. Millikan sprayed a fine mist of oil-drops into an evacuated chamber using an atomizer. The chamber has two metal plates, which are charged with high voltage. Some of the oil drops find their way into the region between these two plates through a pin-hole on the top plate. In this region they move under the action of gravity and air resistance. If these drops are exposed to an ionizing radiation from an X-ray source, some of electrons of the ionized air attach themselves to the oil drops, making these drops negatively charged. The droplets are illuminated by a light source at right angles to a viewing microscope. With careful switching of the voltage, the droplet can be observed for a long time. Initially, the plates are uncharged. A droplet acquires a terminal speed $v_0$

as it falls between the plates under the action of gravity and air resistance.

$$\text{Air Resistance} = \frac{6\pi \eta a v_0}{M g} \quad \text{(Stoke's Law)}$$

where $a$ is the radius of the droplet. You can determine the terminal velocity by observing the time of fall of the droplet.

$$M = 4\pi a^3 \rho / 3$$

If density of oil is $\rho$, the mass $M$.

Using these, mass and the radius of the drop can be determined:

$$a = \sqrt[3]{\frac{9\pi \rho v_0}{2g \rho}}$$
Once the mass is determined, the droplet is subjected to the electric field such that the droplet starts moving upward. In this situation, the air resistance (which is opposite to velocity direction) as well as the weight of the droplet balances the electric force and the droplet acquires a new terminal speed $v$.

$$qE = mg + 6\pi \eta av$$

$$q = \frac{6\pi \eta a}{E} (v + v_0)$$

By measuring the time of fall between two positions the terminal velocity is determined. It is found that the charge on the droplet is an integral multiple of an elementary charge. The value of the elementary charge determined by Millikan was $1.592 \times 10^{-19}$ coulomb.
The currently accepted value of the charge of an electron is \(-1.602 \times 10^{-19}\) C.

The experiment shows:

**Electric charge exists in basic natural units and the value of the basic unit is equal to the magnitude of the charge of an electron**

**Exercise 1**

In a Millikan experiment with a plate separation of 1.6 cm, the charge on an oil drop of mass \(1.0 \times 10^{-15}\) kg could be altered by switching on and off the radiation. In each case, the droplet could be suspended motionless in the chamber by application of voltage between the as plates given in the table below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>391.5</td>
</tr>
<tr>
<td>2</td>
<td>407.8</td>
</tr>
<tr>
<td>3</td>
<td>376.4</td>
</tr>
<tr>
<td>4</td>
<td>337.5</td>
</tr>
<tr>
<td>5</td>
<td>362.5</td>
</tr>
</tbody>
</table>

Using the above data estimate the least quantum of charge on the droplet.
Magnetic Monopoles

Though the electric charge is always quantized, isolated magnetic charge (i.e. isolated magnetic north or South Pole) has never been experimentally detected. Magnetic poles always appear in pairs, though there is no apparent reason for the same. This creates an asymmetry between the electric and magnetic phenomena and has remained a puzzle for the scientists. If magnetic monopoles are found, a similar quantization of their magnetic charge is predicted by physicists.

CONSERVATION OF CHARGE

In any physical process involving charged particles, charge cannot be destroyed; it can only be transferred from one particle to another. No reaction has ever been found where the total charge of reactants is different from that of the products. If $Q_1$ and $Q_2$ are charges of two interacting bodies, which result in two products carrying charges $Q'_1$ and $Q'_2$, then

$$Q_1 + Q_2 = Q'_1 + Q'_2$$

where the addition is algebraic. For instance, when an electron ($e^-$), with charge $-|e|$ and its anti-particle positron ($e^+$) with charge $|e|$ annihilate to produce two photons ($\gamma$),

$$e^+ + e^- = 2\gamma$$

it does not violate *charge conservation* because photon is charge neutral.

Exercise 2

Which of the following equations are inconsistent with charge conservation?

$$^{238}U \rightarrow ^{234}Th + ^4He$$

$$\gamma \rightarrow e + e$$

$$^2H + ^2H \rightarrow ^3He + n$$
COULOMB's LAW

Coulomb's law is the fundamental law of electromagnetism. It gives the force between two charges $Q_1$ and $Q_2$, separated by a distance $r$. The features of the force are:

- proportional to the product of the charges (like charges repel and unlike charges attract),
- inversely proportional to the square of the distance between them,
- is central in nature

Mathematically, the force on $Q_2$ due to $Q_1$ is a unit vector from $Q_1$ to $Q_2$. We can write

$$F_{21} \propto \frac{Q_1 Q_2}{r^2} \hat{r}$$

$\hat{r}$ is a unit vector from $Q_1$ to $Q_2$. 

$\varepsilon_0$ is called permittivity of free space. In SI units, its value is

$$\varepsilon_0 = 8.85 \times 10^{-12} \, \text{coul}^2 \text{N.m}^{-2}$$

Approximately,

$$\frac{1}{4\pi \varepsilon_0} \approx 8.99 \times 10^9 \, \frac{\text{N.m}^2}{\text{coul}^2}$$
By Newton's third law, the force on due to is , SUPERPOSITION PRINCIPLE
According to superposition principle, the force on a charge (called the test charge) due to a
collection of charges is equal to the vector sum of forces due to each charge on the test charge.
A test charge, by definition, is a charge with infinitesimally small magnitude such that other
charges in its vicinity experience negligible force due to it. The force on the test charge due
to charges \( Q_1, Q_2, \ldots, Q_N \) is

\[
F = \frac{1}{4\pi \varepsilon_0} \left( \frac{Q_1 Q}{r_1^2} \hat{r}_1 + \frac{Q_2 Q}{r_2^2} + \ldots \right)
\]

\[
= \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N} \frac{Q_i Q}{r_i^2} \hat{r}_i
\]

**Example 1:**

Two positive charges 2 nC each are placed along the x-axis at \((0, 4)\) m. A third charge of opposite
sign with magnitude 3 nC is placed on the y-axis at \((0, 4)\) m. Find the net force on the charge on
the y-axis due to the other two charges.

**Solution:**
The forces are attractive and are shown. The magnitude of each force is

\[ F_1 = F_2 = \frac{1}{4\pi \varepsilon_0} \frac{3 \times 10^{-9} \cdot 2 \times 10^{-9}}{5^2} = \frac{(54/25)}{10^{-9}} \text{ N} \]

\[ \sin^{-1}(3/5) \]

The angle \( \theta \) is \( \sin^{-1}(3/5) \). Clearly, \( F_{1x} = -F_{2x} \) and the x-components cancel.

The net force is towards the origin along the y-axis and is

\[ F_{net} = F_{1y} + F_{2y} = (F_1 + F_2) \cos \theta = 3.46 \times 10^{-9} \text{ N} \]

**Exercise**

Four point charges, each \( +4 \text{n C} \) are placed one each at the four corners of a square of side 0.12m. Find the force on one of the charges due to three others.

\[ 1.9 \times 10^{-5} \text{ N} \]

[Ans.: along line joining the particular charge to the charge at the diagonally opposite corner, directed outward from the square.]
**ELECTRIC FIELD:** A field, in general, is a physical quantity, which is specified at all points in space. For instance, one could talk of a temperature field, which is described by specifying the temperature at all points of space at a given time. This is an example of a scalar field, as the physical quantity involved, the temperature, is a scalar. The region of space around a charged object is said to be the Electric Field of the charged object. If a test charge is brought into this region, it experiences a force. The electric field at a point is defined as the force experienced by a unit positive test charge kept at that point. The direction of the electric field is the direction of the force on such a positive test charge. The electric field $\vec{E}$, therefore, is a vector field.

$$\vec{E} = \frac{\vec{F}}{q}$$

where $\vec{F}$ is the force experienced by a charge $q$.

Coulomb's law allows us to calculate the field due to a charge $Q$. The force on a test charge due to charge $Q$ is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{r^2} \hat{r}$$

where $\hat{r}$ is a unit vector joining $Q$ and $q$, in a direction as defined above. The electric field, which is the force per unit charge is therefore, given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$$

The unit of the electric field is Newton/coulomb. The field depends only on $Q$ and position vector of any point on space with respect to the position of the charge $Q$. Actual presence of a test charge is not required. The field is defined at every point in space.

**Example 2:**

Find the electric field at a distance $d$ from the midpoint of a finite charged rod of length $L$ carrying a charge $Q$.

**Solution:**
Choose the coordinates as shown. Let the rod be taken along the y-axis with the origin at the centre. Divide the rod into small segments of length \( dy \). The field produced by the element at a height \( y \) is

\[
dE = \frac{1}{4\pi \varepsilon_0} \frac{Q dy}{L} \frac{1}{(x^2 + y^2)}
\]

is directed along \( \vec{AP} \) in the figure.

**Electric Lines of Force**

Electric lines of force (also known as *field lines*) is a pictorial representation of the electric field. These consist of directed lines indicating the direction of electric field at various points in space.

- There is no rule as to how many lines are to be shown. However, it is customary to draw number of lines proportional to the charge. Thus if \( N \) number of lines are drawn from \( Q \) or into a charge \( 2Q \), \( 2N \) number of lines would be drawn for charge \( Q \).

- The electric field at a point is directed along the tangent to the field lines. A positive charge at this point will move along the tangent in a direction indicated by the arrow.

- Lines are dense close to a source of the electric field and become sparse as one moves away.

- Lines originate from a positive charge and end either on a negative charge or move to infinity. Lines of force due to a solitary
Negative charge is assumed to start at infinity and end at the negative charge.

Field lines do not cross each other. (If they did, the field at the point of crossing will not be uniquely defined.)

A neutral point is a point at which field strength is zero. This occurs because of cancellation of electric field at such a point due to multiple charges.

**Exercise**

Draw field lines and show the neutral point for a charge located at \((1,0)\), and \(-Q\) located at \((-1,0)\).
Electric Flux

Objectives
In this lecture you will learn the following

- Concept of flux and calculation of electric flux through simple geometrical objects
- Gauss's Law of electrostatics
- Applications of Gauss's Law to Calculate electric field due to a few symmetric charge distributions.

Electric Flux

The concept of flux is borrowed from flow of water through a surface. The amount of water flowing through a surface depends on the velocity of water, the area of the surface and the orientation of the surface with respect to the direction of velocity of water. Though an area is generally considered as a scalar, an element of area may be considered to be a vector because

- It has magnitude (measured in m$^2$).
- If the area is infinitesimally small, it can be considered to be in a plane. We can then associate a direction with it. For instance, if the area element lies in the x-y plane, it can be considered to be directed along the z-direction. (Conventionally, the direction of the area is taken to be along the outward normal.)
In the figure above, the length of the vector \( \vec{S} \) is chosen to represent the area in some convenient unit and its direction is taken to be along the outward normal to the area.

We define the flux of the electric field through an area \( d\vec{S} \) to be given by the scalar product

\[
d\phi = \vec{E} \cdot d\vec{S}
\]

If \( \theta \) is the angle between the electric field and the area vector \( d\phi = |\vec{E}| |d\vec{S}| \cos \theta \)

For an arbitrary surface \( S \), the flux is obtained by integrating over all the surface elements

\[
\phi = \int d\phi = \int_S \vec{E} \cdot d\vec{S}
\]

If the electric field is uniform, the angle \( \theta \) is constant and we have

\[
\phi = ES \cos \theta = E (S \cos \theta)
\]

Thus the flux is equal to the product of magnitude of the electric field and the projection of area perpendicular to the field.

Unit of flux is \( \text{N-m}^2/\text{C} \). Flux is positive if the field lines come out of the surface and is negative if they go into it.
Solid Angle: The concept of solid angle is a natural extension of a plane angle to three dimensions. Consider an area element dS at a distance r from a point P. Let be the unit vector along the outward normal to dS. The element of the solid angle subtended by the area element at P is defined as

$$d\Omega = \frac{dS_{\perp}}{r^2}$$

where is the projection of along a direction perpendicular to . If is the angle between and , then,

$$d\Omega = \frac{dS \cos \alpha}{r^2}$$

Solid angle is dimensionless. However, for practical reasons it is measured in terms of a unit called steradian (much like the way a planar angle is measured in terms of degrees). The maximum possible value of solid angle is , which is the angle subtended by an area which encloses the point P completely.

Example
A right circular cone has a semi-vertical angle . Calculate the solid angle at the apex P of the cone.

Solution:
The cap on the cone is a part of a sphere of radius R, the slant length of the cone. Using spherical polar coordinates, an area element on the cap is $R^2 \sin \theta d\theta d\phi$, where $\theta$ is the polar angle and $\phi$ is the azimuthal angle.

Here, $\phi$ goes from 0 to $2\pi$ while $\theta$ goes from 0 to $\alpha$. Thus the area of the cap is
Thus the solid angle at P is
\[ d\Omega = \frac{dA}{R^2} = 2\pi (1 - \cos \alpha) \]

Exercise

Calculate the solid angle subtended by an octant of a sphere at the centre of the sphere.

(Ans. \( \pi / 2 \))

The flux per unit solid angle is known as the intensity.

Example 3
An wedge in the shape of a rectangular box is kept on a horizontal floor. The two triangular faces and the rectangular face ABFE are in the vertical plane. The electric field is horizontal, has a magnitude \( 8 \times 10^4 \) N/C and enters the wedge through the face ABFE, as shown. Calculate the flux through each of the faces and through the entire surface of the wedge.
Solution:
The outward normals to the triangular faces AED, BFC, as well as the normal to the base are perpendicular to \( \vec{E} \). Hence the flux through each of these faces is zero. The vertical rectangular face ABFE has an area 0.06 \( m^2 \). The outward normal to this face is perpendicular to the electric field. The flux is entering through this face and is negative. Thus flux through ABFE is

\[
\phi_1 = -0.06 \times 8 \times 10^4 = -4.8 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}
\]

To find the flux through the slanted face, we need the angle that the normal to this face makes with the horizontal electric field. Since the electric field is perpendicular to the side ABFE, this angle is equal to the angle between AE and AD, which is \( \cos^{-1}(\frac{3}{5}) \). The area of the slanted face ABCD is 0.1 \( m^2 \). Thus the flux through ABCD is

\[
\phi_2 = 0.1 \times 8 \times 10^4 \times \left(\frac{3}{5}\right) = +0.48 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}
\]

The flux through the entire surface of the wedge is

\[\phi_1 + \phi_2 = 0\]

Example 4:
Calculate the flux through the base of the cone of radius \( R \).
Solution:
The flux entering is perpendicular to the base. Since the outward normal to the circular base is in the opposite sense, the flux is negative and is equal to the product of the magnitude of the field and the area of the base, the flux, therefore is, \( \pi R^2 E \)

Example 5:
Calculate the flux coming out through the curved surface of the cone in the above example.

Solution:

Consider a circular strip of radius \( r \) at a depth \( h \) from the apex of the cone. The angle between the electric field through the strip and the vector \( \vec{dS} \) is \( \pi - \theta \), where \( \theta \) is the semi-angle of the cone. If \( dl \) is the length element along the slope, the area of the strip is \( 2\pi r dl \). Thus,

\[
\vec{E} \cdot \vec{dS} = 2\pi r dl \mid E \mid \sin \theta
\]
\[ l = \frac{h}{\cos \theta} \quad dl = \frac{dh}{\cos \theta} \]

We have, so that. Further, \( r = h \tan \theta \)

Substituting, we get

\[ \vec{E} \cdot d\vec{S} = 2\pi h \tan^2 \theta \mid E \mid dh \]

Integrating from \( h = 0 \) to \( h = H \), the height of the cone, the outward flux is

\[ \int e^{\pi R^2} \tan^2 \theta = \pi R^2 \mid E \mid \]

**Example 6:**

A charge \( Q \) is located at the center of a sphere of radius \( R \). Calculate the flux going out through the surface of the sphere.

By Coulomb's law, the field due to the charge is radial and is given on the surface of the sphere by,

\[ \vec{E} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{R^2} \hat{r} \]

The direction of the area vector \( d\vec{S} \), is also radial at each point of the surface \( d\vec{S} = dS \hat{r} \). The flux

\[ \phi = \int \vec{E} \cdot d\vec{S} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{R^2} \int dS \]
The integral over \( dS \) is equal to the surface area of the sphere, which is, \( 4\pi R^2 \). Thus the flux out of the surface of the sphere is

\[
\phi = \frac{1}{4\pi \epsilon_0} \frac{Q}{R^2} \cdot 4\pi R^2 = \frac{Q}{\epsilon_0}
\]

**GAUSS'S LAW - Integral form**

The flux calculation done in Example 4 above is a general result for flux out of any closed surface, known as Gauss's law.

Total outward electric flux \( \phi \) through a closed surface \( \mathcal{S} \) is equal to \( \frac{1}{\epsilon_0} \) times the charge enclosed by the volume defined by the surface \( \mathcal{S} \).

Mathematically, the surface integral of the electric field over any closed surface is equal to the net charge enclosed divided by \( \epsilon_0 \).
The law is valid for arbitrary shaped surface, real or imaginary.

Its physical content is the same as that of Coulomb's law.

In practice, it allows evaluation of electric field in many practical situations by forming imagined surfaces, which exploit symmetry of the problem. Such surfaces are called Gaussian surfaces.

**GAUSS'S LAW - Differential form**

The integral form of Gauss's law can be converted to a differential form by using the divergence theorem. If $V$ is the volume enclosed by the surface $S$,

$$
\int_S \vec{E} \cdot d\vec{S} = \int_V \nabla \cdot \vec{E} \, dv
$$

(A)

If $\rho$ is the volume charge density,

$$
Q = \int_V \rho \, dv
$$

(B)

Thus we have

$$
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}
$$

**Applications of Gauss's Law**

Field due to a uniformly charged sphere of radius with a charge $\lambda$

Gaussian surface is a cylinder of radius $r$ and length $L$.

By symmetry, the field is radial. Gaussian surface is a concentric sphere of radius $r$. The outward normals to the Gaussian surface are parallel to the field at every point. Hence For,

so that

The field outside the sphere is what it would be if all the charge is concentrated at the origin of the sphere.

For, a fraction of the total charge is enclosed within the Gaussian surface, so that
The field inside is

\[ \oint \vec{E} \cdot d\vec{S} = |E| \cdot 2\pi r L \]

\[ = \frac{Q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} \]

Thus

\[ \vec{E} = \frac{\lambda}{2\pi \varepsilon_0 r} \hat{\rho} \]

where \( \hat{\rho} \) is a unit vector perpendicular to the line, directed outward for positive line charge and inward for negative line charge.

**Field due to an infinite charged sheet with surface charge density \( \sigma \)**

Choose a cylindrical *Gaussian pillbox* of height \( h \) (with \( h/2 \) above the sheet and \( h/2 \) below the sheet) and radius \( r \).
The amount of charge enclosed is area times the surface charge density, i.e., \( Q = \pi r^2 \sigma \). By symmetry, the field is directed perpendicular to the sheet, upward at points above the sheet and downward for points below. There is no contribution to the flux from the curved surface. The flux from the two end faces is \( \pi r^2 \left| \mathbf{E} \right| \) each,

\[
2\pi r^2 \left| \mathbf{E} \right| = \frac{Q}{\varepsilon_0} = \frac{\pi r^2 \sigma}{\varepsilon_0}
\]

so that

\[
\mathbf{E} = \frac{\sigma}{2\varepsilon_0} \hat{n}
\]

where \( \hat{n} \) is a unit vector perpendicular to the sheet, directed upward for points above and downwards for points below (opposite, if the charge density is negative).

**Field due to a uniformly charged sphere of radius \( R \) with a charge \( Q \)**

By symmetry, the field is radial. Gaussian surface is a concentric sphere of radius \( r \). The outward normals to the Gaussian surface is parallel to the field \( \mathbf{E} \) at every point. Hence

\[
\int \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 \left| \mathbf{E} \right|
\]
\[ r > R \]

For \[ r > R \]
so that
\[ 4\pi r^2 \mid E \mid = \frac{Q}{r^2} \]

The field outside the sphere is what it would be if all the charge is concentrated at the origin of the sphere.

\[ r < R \quad \frac{r^3}{R^3} \]

For \[ r < R \quad \frac{r^3}{R^3} \], a fraction of the total charge is enclosed within the Gaussian surface, so that

\[ 4\pi r^2 \mid E \mid = \frac{1}{\varepsilon_0} \frac{Qr^3}{R^3} \]

The field inside is

\[ \vec{E} = \frac{Q}{4\pi \varepsilon_0 \frac{r}{R^3}} \hat{r} \]
Properties of Conductors

Objectives
In this lecture you will learn the following

- Properties of Conductors in equilibrium
- Induced charges in a conductor
- Electrostatic Shielding
- Concept of electric potential and its calculation
- Determination of electric field from a knowledge of potential

Properties of Conductors

A conductor (typically, a metal or ionic conductors like HCl or NaCl dissolved in water) allows free movement of charges,

- **The electric field inside a conductor is zero.** In an equilibrium situation, there cannot be an electric field inside a conductor as this would cause charges (electrons or ions) to move around.

- **Charge density inside a conductor is zero.** This follows from Gauss's law

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \]

As \( \vec{E} = 0 \), the charge density \( \rho = 0 \).

(This does not suggest that there is no charge inside only that the positive and negative charges cancel inside a conductor.)

- **Free charges exist only on the surface of a conductor.** Since there is no net charge inside, free charges, if any, have to be on the surface.

- **At the surface of a conductor, the electric field is normal to the surface.** If this were not so, the charges on the surface would move along the surface because of the tangential component of the field, disturbing equilibrium.
Induced Charges in a conductor:

The above properties of a conductor influence the behavior of a conductor placed in an electric field. Consider, for instance, what happens when a charge $+q$ is brought near an uncharged conductor. The conductor is placed in the electric field of the point charge. The field inside the conductor should, however, be zero. This is achieved by a charge separation within the conductor which creates its own electric field which will exactly compensate the field due to the charge $+q$. The separated charges must necessarily reside on the surface.

Another way of looking at what is happening is to think of the free charges in the conductor being attracted towards (or repelled from) the external charge. Thus the surface of the conductor towards the external charge is oppositely charged. To keep the charge neutrality, the surface away from the external charge is similarly charged.

Example 9:

A charge $Q$ is located in the cavity inside a conducting shell. In addition, a charge $2Q$ is distributed in the conducting shell. Find the distribution of charge in the shell.
Take a Gaussian surface entirely within the conducting shell, completely enclosing the cavity. Everywhere on the Gaussian surface $\mathbf{E} = 0$. The flux and therefore, the charge enclosed is zero within the Gaussian surface. As the cavity contains a charge $Q$, the inner surface of the cavity must have charge $-Q$. As the conductor has distributed charge $2Q$, the charge on the outside surface is $3Q$.

The electric field outside the shell is same as that due to a point charge $+3Q$ located at the centre of the sphere. The reason is that the charge in the cavity is compensated exactly by the induced charges in the surface of the cavity. Thus the effect of the charge $Q$ is not felt at points outside the sphere. The uniformly distributed charge of $3Q$ over the surface of the sphere gives the same field as that of a point charge located at the centre.

**Exercise**

Consider the system of conductors shown with two cavities. A charge $+Q$ is kept at the center. (i) Determine the charge distributions on the surfaces marked 1, 2, 3 and 4, (ii) Is the potential of surface 1 lower, higher or same as that of surface 2 ? (iii) Is the potential of surface 4 lower, higher or same as that of surface 1 ? (Answer: (i) $-Q$ for 1 and 3, (ii) $+Q$ for 2 and 4 (ii) equal (iii) lower.)
Example 10:

Calculate the electric field outside a conductor carrying a surface charge density $\sigma$.

Take a Gaussian pillbox in the shape of a cylinder of height $h$ with $h/2$ inside and $h/2$ outside the conductor. Let the cross sectional area be $dS$ normal to the surface. The electric field is normal to the surface. As the field inside is zero and there is no tangential component of the field at the surface, the flux goes out only through the outer cap of the cylinder. The charge enclosed is $\sigma dS$ and the flux is $EdS$. The electric field is normal to the surface.

Applying Gauss's law

$$E = \frac{\sigma}{\epsilon_0} \hat{n}$$

Exercise 1:

Two parallel, infinite plates made of material of perfect conductor, carry charges $Q_1$ and $Q_2$. The plates have finite thickness. Show that the charge densities on the two adjacent inside surfaces are equal and opposite while that on the two outside surfaces are equal.

(Hint: Field inside the plates due to four charged surfaces must be zero.)
**ELECTROSTATIC POTENTIAL**

Electrostatic force is a conservative force, i.e., the work done by the force in moving a test charge from one point to another is independent of the path connecting the two points.

**Unit of Potential**

Since potential is the energy per unit charge, the unit of potential is Joule/Coulomb, which is called a volt. The unit of the electric field which we have so far been using as Newton/Coulomb is more commonly referred as volt/meter.

**Potential Function satisfies Superposition Principle**

Consider a collection of charges $Q_1, Q_2 \ldots$. The electric field at a point due to the distribution $E_i$ of charges obeys superposition principle. If $E_i$ is the electric field at a point P due to the charge $Q_i$, the net electric field at P is

$$E = \sum_i E_i$$

The potential $\phi$ at the point P (with respect to the reference point $P_0$) is

$$\phi(P) = \int_{P_0}^P E \cdot dl = \int_{P_0}^P \sum_i E_i \cdot dl = \sum_i \int_{P_0}^P E_i \cdot dl = \sum_i \phi_i(P)$$

where $\phi_i(P)$ is the potential at P due to the charge $Q_i$.

**Determining Electric Field from knowledge of Potential**

The potential at the position $\vec{x}$ is given by the expression

$$\phi(\vec{x}) = -\int_{\vec{x}_0}^{\vec{x}} E \cdot d\vec{l} \quad \text{(A)}$$

where $\vec{x}_0$ is a reference point such that $\phi(\vec{x}_0) = 0$.

In one dimension,
\( \phi(x) = \int_{x_0}^{x} E(x) \, dx \)

Differentiate both sides with respect to the upper limit of integration, i.e. \( x \)

\[
\frac{\partial \phi}{\partial x} = -E(x)
\]

In three dimensions, we use the fundamental theorem on gradients

\[
\phi(\vec{x}) - \phi(\vec{x}_0) = \int_{\vec{x}_0}^{\vec{x}} (\vec{\nabla} \phi) \cdot d\vec{l}
\]

which gives

\[
\phi(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} (\vec{\nabla} \phi) \cdot d\vec{l}
\]

Comparing the above with eqn. (A) above,

\[
\vec{E} = -\nabla \phi
\]

In Cartesian coordinates,

\[
\vec{E} = -\hat{i} \frac{\partial \phi}{\partial x} - \hat{j} \frac{\partial \phi}{\partial y} - \hat{k} \frac{\partial \phi}{\partial z}
\]

**Electric Field is Irrotational, i.e. Curl \( \vec{E} = 0 \)**

This follows from Stoke's theorem.

\[
\oint \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}
\]

where the surface integral is over *any* surface bounded by the closed curve. As the surface \( S \) is arbitrary (as long as it is bounded by the same curve), the integrand must vanish. Hence,
Potential and Potential Energy

Potential and Potential energy are different, though they are related. Electric potential at a point is the potential energy of a unit test charge kept at that point.

Potential is the property of the field and is defined at every point, whether or not a charge is located at the point. It is the potential energy that the unit test charge would have if it happens to be located at that point.

The potential energy of a positive charge $q$ increases if it is taken to a region of higher potential. If electric force is the only force acting on the particle, its kinetic energy would decrease by a similar amount.

Let the charge have a velocity $\mathbf{v}_1$ at the position $P_1$ where the electrostatic potential is $\phi_1$. If it moves to a position $P_2$, where the potential is $\phi_2$, then, the velocity $\mathbf{v}_2$ of the particle at this point is given by the energy conservation.
The work done on the particle in moving from a potential \( \phi_1 \) to the potential \( \phi_2 \) is given by the "work-energy theorem"

\[
\frac{1}{2}m v_1^2 + q\phi_1 = \frac{1}{2}m v_2^2 + q\phi_2
\]

Volt, the unit of potential difference, may be interpreted as follows. If a charge of one coulomb moves through a potential difference such that in the new position the potential is lower by 1 Volt, the kinetic energy of the charge increases by 1 Joule.

**Electron Volt**

In atomic and nuclear physics, a commonly used unit of energy is *electron volt*. An electron volt is the change in the kinetic energy of an electron when it is taken through a potential difference of one volt. Thus,

\[
1\text{eV} = |\text{charge of an electron}| \times 1\text{volt} = 1.6 \times 10^{-19}\text{Coulomb} \times 1\text{volt} = 1.6 \times 10^{-19}\text{ Joule}
\]
Electrostatic Potential

Objectives

In this lecture you will learn the following

✓ Electric Dipole and field due to a dipole
✓ Torque on a dipole in an inhomogeneous electric field
✓ Potential Energy of a dipole
✓ Energy of a system of charges - discrete and continuous

Potential and Field due to an Electric Dipole

An electric dipole consists of two equal and opposite charges $+q$ and $-q$ separated by a small distance $a$.

The Electric Dipole Moment $\vec{p}$ is defined as a vector of magnitude $qa$ with a direction from the negative charge to the positive charge. In many molecules, though the net charge is zero, the nature of chemical bonds is such that the positive and negative charges do not cancel at every point. There is a small separation between the positive charge centers and negative charge centres. Such molecules are said to be polar molecules as they have a non-zero dipole moment. The figure below shows an asymmetric molecule like water which has a dipole moment $6.2 \times 10^{-30}$ C-m.
In the polar coordinates shown in the figure
\[ \vec{p} = p \cos \theta \hat{r} - p \sin \theta \hat{\theta} \]

where \( \hat{r} \) and \( \hat{\theta} \) are unit vectors in the radial and tangential directions, taken respectively, in the direction of increasing \( r \) and increasing \( \theta \). The electric potential at a point \( P \) with a position vector \( \vec{r} \) is

\[
\phi(\vec{r}) = \frac{1}{4\pi \epsilon_0} \left[ \frac{q}{r_1} - \frac{q}{r_2} \right] = \frac{q}{4\pi \epsilon_0} \frac{r_2 - r_1}{r_1 r_2}
\]
If the distance \( a \) is small compared to \( r \) (i.e., if the point \( P \) is far away from the dipole), we may use

\[
\mathbf{r}_2 - \mathbf{r}_1 \approx a \cos \theta \quad r_1 r_2 \approx r^2
\]

where \( \theta \) is the angle between \( \mathbf{r} \) and the dipole moment vector \( \mathbf{p} \). This gives

\[
\phi(\mathbf{r}) \approx \frac{qa \cos \theta}{4\pi\varepsilon_0 r^2} = \frac{p \cos \theta}{4\pi\varepsilon_0 r^2}
\]

Electric Field of a Dipole

A. CARTESIAN COORDINATES

It is convenient to define the Cartesian axes in the following way. Let the dipole moment vector be taken along the \( z \)-axis and position vector \( \mathbf{r} \) of \( P \) in the \( y-z \) plane (We have denoted the point \( \mathbf{p} \) where the electric field is calculated by the letter \( P \) and the electric dipole moment vector as \( \mathbf{p} \)).

\[
\cos \theta = \frac{z}{r} \quad r = \sqrt{y^2 + z^2}
\]

We then have \( \cos \theta \). Thus

\[
\phi(x, y, z) = \frac{p \cos \theta}{4\pi\varepsilon_0 r^2} = \frac{pz}{(y^2 + z^2)^{3/2}}
\]
\[ E_x = 0 \]

Since \( \phi \) is independent of \( x \), the y and z components are

\[
E_y = -\frac{\partial}{\partial y} \left[ \frac{pz}{4\pi \epsilon_0 (y^2 + z^2)} \right] = \frac{3pzy}{4\pi \epsilon_0 (y^2 + z^2)^{5/2}} = \frac{3p}{4\pi \epsilon_0} \frac{yz}{r^5}
\]

and

\[
E_z = -\frac{\partial}{\partial z} \left[ \frac{pz}{4\pi \epsilon_0 (y^2 + z^2)} \right] = \frac{-p}{4\pi \epsilon_0} \frac{1}{(y^2 + z^2)} + \frac{3pz^2}{4\pi \epsilon_0 (y^2 + z^2)^{5/2}} = \frac{p}{4\pi \epsilon_0} \frac{2z^2 - y^2}{r^5}
\]

B. POLAR COORDINATES

\( r - \theta \)

In polar coordinates, the radial and tangential components of the field are as follows:

\[
E_r = -\frac{\partial \phi}{\partial r} = \frac{2p \cos \theta}{4\pi \epsilon_0 r^3}
\]

\[
E_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{p \sin \theta}{4\pi \epsilon_0 r^3}
\]

GENERAL EXPRESSION

A representation independent form for the dipole field can be obtained from the above. We have

\[
\vec{E} = E_r \hat{r} + E_\theta \hat{\theta} = \frac{p}{4\pi \epsilon_0 r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right]
\]

\[
\vec{p} = p \cos \theta \hat{r} - p \sin \theta \hat{\theta}
\]

Using \( \vec{p} \), we get
This form does not depend on any particular coordinate system. Note that, at large distances, the dipole field decreases with distance as \( \frac{1}{r^3} \) while monopole field (i.e. field due to a point charge) decreases as \( \frac{1}{r^2} \).

**Dipole in a uniform Electric Field**

The net force on the dipole is zero. There is net torque acting on the dipole. If \( \alpha \) is the length of the dipole, the torque is

\[
\tau = (qE) \times a \sin \theta = p E \sin \theta
\]

Expressing in vector form,

\[
\vec{\tau} = \vec{p} \times \vec{E}
\]

If \( \theta = 0^\circ \) or \( 180^\circ \), (i.e. when the dipole is aligned parallel or antiparallel to the field) the torque vanishes and the dipole is in equilibrium. The equilibrium is stable if \( \theta = 0^\circ \) and unstable if \( \theta = 180^\circ \).

**Work done in turning a dipole from equilibrium**

If the dipole is twisted by an angle \( \theta \) from its stable equilibrium position, work has to be done by the external agency.
This work becomes the potential energy of the dipole in this position.

Energy of a Dipole

To calculate energy of a dipole oriented at an angle $\theta$ in the electric field, we have to add to the work done above, the energy of the dipole in the equilibrium position. This is equal to the work done in bringing the dipole from infinity to the equilibrium position. The dipole may be aligned in the direction of the field at infinity without any cost of energy. We may now displace the dipole parallel to the field to bring to the equilibrium position. As the negative charge is displaced along the field by an additional distance $a$, the work done is $-qEa = -pE$, which is the potential energy of the dipole in equilibrium.

The potential energy of the dipole at position $\theta$ is

$$\mathcal{E} = -pE + pE(1 - \cos \theta) = -pE \cos \theta = -\vec{p} \cdot \vec{E}$$

The energy is positive if $\theta$ is acute and is negative if $\theta$ is obtuse.

Potential Energy of a System of Charges

Assume all charges to be initially at infinity. We assemble the charges by bringing the charges one by one and fix them in their positions. There is no energy cost in bringing the first charge $q_1$ and putting it at $P$, as there is no force field. Thus

$$W_1 = 0$$
We now bring the second charge and take it to point $P^2$. Since this charge moves in the potential field of the first charge, the work done in bringing this charge is

$$W_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} = q_2 \phi_1 (P_2)$$

where $\phi_1 (P_2)$ is the potential at $P^2$ due to the charge at $P^1$.

The third charge $q_3$ is to be brought to $P^3$ under the force exerted by $\frac{q_1}{r_{13}}$ and $\frac{q_2}{r_{23}}$ and is

$$W_3 = \frac{1}{4\pi\varepsilon_0} \left( \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) = q_3 \left( \phi_1 (P_3) + \phi_2 (P_3) \right)$$

and so on.

The work done in assembling $N$ charges $q_1, q_2, \ldots, q_N$, located respectively at $r_1, r_2, \ldots, r_N$ is
\[ W = W_1 + W_2 + \ldots + W_N \]
\[ = \frac{1}{4\pi \varepsilon_0} \sum_{i<j}^{N} \sum_{j=2}^{N} \frac{q_i q_j}{r_{ij}} \]
\[ = \frac{1}{8\pi \varepsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{q_i q_j}{r_{ij}} \]

The extra factor of \( \frac{1}{2} \) in the last expression is to ensure that each pair \((i,j)\) is counted only once. The sum excludes the terms \( i = j \). Since the potential at the \( i \)-th position due to all other charges is

\[ \phi(r_i) = \frac{1}{4\pi \varepsilon_0} \sum_{j \neq i} \frac{q_i}{r_{ij}} \]

we get

\[ W = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(i) \]

**Energy of a continuous charge distribution**

If \( \rho(\vec{r}) \) is the density of charge distribution at \( \vec{r} \), we can generalize the above result

\[ W = \frac{1}{8\pi \varepsilon_0} \int \int \rho(\vec{r}) \rho(\vec{r}') \frac{d\tau d\tau'}{\mid \vec{r} - \vec{r}' \mid} \]
\[ = \frac{1}{2} \int \rho(\vec{r}) \phi(\vec{r}) d\tau \]

(In case of a line charge or a surface charge distribution, the integration is over the appropriate dimension).

Since the integral is over the charge distribution, it may be extended over all space by defining the charge density to be zero outside the distribution, so that the contribution to the integral comes only from the region of space where the charge density is non-zero. Writing

\[ W = \int_{\text{all space}} \rho(\vec{r}) \phi(\vec{r}) d\tau \]

From the differential form of Gauss's law, we have
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

With this
\[ W = \frac{\varepsilon_0}{2} \int_{\text{all space}} (\nabla \cdot \vec{E}) \phi d\tau \]

On using the vector identity
\[ \nabla \cdot (\phi \vec{E}) = \phi \nabla \cdot \vec{E} + \vec{E} \cdot \nabla \phi \]

\[ \vec{E} = -\nabla \phi \]

we get, using
\[ W = \frac{\varepsilon_0}{2} \int_{\text{surface}} \nabla \cdot (\phi \vec{E}) d\tau + \frac{\varepsilon_0}{2} \int_{\text{all space}} |\vec{E}|^2 d\tau \]

The first integral can be converted to a surface integral by using divergence theorem and the surface can be taken at infinite distances, where the electric field is zero. As a result the first integral vanishes and we have
\[ W = \frac{\varepsilon_0}{2} \int_{\text{all space}} |\vec{E}|^2 d\tau \]
Capacitance

Objectives

In this lecture you will learn the following

✓ Capacitors in series and in parallel
✓ Properties of dielectric
✓ Conductor and dielectric in an electric field
✓ Polarization and bound charges
✓ Gauss's Law for dielectrics

Capacitors in Combination:

Capacitors can be combined in series or parallel combinations in a circuit.

Parallel Combination

When they are in parallel, the potential difference across each capacitor is the same.
The charge on each capacitor is obtained by multiplying with the capacitance, i.e.
\[ Q_1 = C_1 V \quad Q_2 = C_2 V \ldots \]
Since total charge in the capacitors is sum of all the charges, the effective capacitance of the combination is
\[ C = \frac{Q_1 + Q_2 + Q_3 + \ldots}{V} = \frac{Q_1}{V} + \frac{Q_2}{V} + \frac{Q_3}{V} + \ldots = C_1 + C_2 + C_3 + \ldots \]

**Series Combination:**

When capacitors are joined end to end in series, the first capacitor gets charged and induces an equal charge on the second capacitor which is connected to it. This in turn induces an equal charge on the third capacitor, and so on.

![Diagram of series combination of capacitors](https://mywbut.com)

The net potential difference between the positive plate of the first capacitor and the negative plate of the last capacitor in series is

The individual voltage drops are
\[ V_1 = \frac{Q_1}{C_1} \quad V_2 = \frac{Q_2}{C_2} \ldots \]

so that
The effective capacitance is, therefore, given by

\[ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \]

**Example:**

Calculate the voltage across the 5 \( \mu \)F capacitor in the following circuit.

The equivalent circuit is shown above. The two 10 \( \mu \)F capacitors in series is equivalent to a 5 \( \mu \)F capacitor.

5 \( \mu \)F in parallel with this equivalent capacitor gives 10 \( \mu \)F as the next equivalent.

The circuit therefore consists of a 10 \( \mu \)F in series with the 20 \( \mu \)F capacitor.

Since charge remains constant in a series combination, the potential drop across the 10 \( \mu \)F capacitor is twice as much as that across 20 \( \mu \)F capacitor. The voltage drop across the 10 \( \mu \)F (and hence across the given 5 \( \mu \)F) is

\[ (2/3) \times 12 = 8 \text{ V}. \]
Exercise

Determine the effective capacitance of the following capacitance circuit and find the voltage across each capacitance if the voltage across the points a and b is 300 V.
Conductors and Dielectric

A conductor is characterized by existence of free electrons. These are electrons in the outermost shells of atoms (the valence electrons) which get detached from the parent atoms during the formation of metallic bonds and move freely in the entire medium in such way that the conductor becomes an equipotential volume.

In contrast, in dielectrics (insulators), the outer electrons remain bound to the atoms or molecules to which they belong. Both conductors and dielectric, on the whole, are charge neutral. However, in case of dielectrics, the charge neutrality is satisfied over much smaller regions (e.g. at molecular level).

Polar and non-polar molecules:

A dielectric consists of molecules which remain locally charge neutral. The molecules may be polar or non-polar. In non-polar molecules, the charge centres of positive and negative charges coincide so that the net dipole moment of each molecule is zero. Carbon dioxide molecule is an example of a non-polar molecule.
In polar molecules, the arrangement of atoms is such that the molecule has a permanent dipole moment because of charge separation. Water molecule is an example of a polar molecule.

When a non-polar molecule is put in an electric field, the electric forces cause a small separation of the charges. The molecule thereby acquires an induced dipole moment. A polar molecule, which has a dipole moment in the absence of the electric field, gets its dipole moment aligned in the direction of the field. In addition, the magnitude of the dipole moment may also increase because of increased separation of the charges.
Conductor in an Electric Field

Consider what happens when a conductor is placed in an electric field, say, between the plates of a parallel plate capacitor.

A polar molecule in an Electric Field

Induced and external electric fields in a conductor

Net field inside a conductor is zero.

As the conductor contains free charges (electrons), these move towards the positive plate, making the surface of the conductor closer to the positive plate of the capacitor negatively...
charged. These are called induced charges. Consequently, the surface of the conductor at the end closer to the negative plate is positively charged. The motion of charges continue till the internal electric field created by induced charges cancel the external field, thereby making the field inside the conductor zero.

**Dielectric in an Electric Field**

A dielectric consists of molecules which may (polar) or may not (non-polar) have permanent dipole moment. Even in the former case, the dipoles in a dielectric are randomly oriented because dipole energies are at best comparable to thermal energy.

When a dielectric is placed in an electric field the dipoles get partially aligned in the direction of the field. The charge separation is opposed by a restoring force due to attraction between the charges until the forces are balanced. Since the dipoles are partially aligned, there is a net dipole moment of the dielectric which opposes the electric field. However, unlike in the case of the conductors, the net field is not zero. The opposing dipolar field reduces the electric field inside the dielectric.

**Dielectric Polarization**

Electric polarization is defined as the dipole moment per unit volume in a dielectric medium. Since the distribution of dipole moment in the medium is not uniform, the polarization $\vec{P}$ is a
function of position. If \( \vec{\rho}(\vec{r}) \) is the sum of the dipole moment vectors in a volume element \( d\tau \) located at the position \( \vec{r} \),

\[
\vec{\rho}(\vec{r}) = \vec{P}(\vec{r}) d\tau
\]

It can be checked that the dimension of \( \vec{P} \) is same as that of electric field divided by permittivity \( \varepsilon_0 \).

Thus the source of polarization field is also electric charge, except that the charges involved in producing polarization are *bound charges*.

**Potential due to a dielectric**

Consider the dielectric to be built up of volume elements \( d\tau \). The dipole moment of the volume element is \( \vec{P} d\tau \). The potential at a point S, whose position vector with respect to the volume element is \( \vec{r} \), is

\[
d\phi = \frac{1}{4\pi\varepsilon_0} \frac{\vec{P} \cdot \hat{r}}{r^2} d\tau
\]

The potential due to the whole volume is

\[
\phi = \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} \frac{\vec{P} \cdot \hat{r}}{r^2} d\tau = \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} \vec{P} \cdot \nabla \left( \frac{1}{r} \right) d\tau
\]

where, we have used
\[ \nabla \left( \frac{1}{r} \right) = \frac{\hat{r}}{r^2} \]

Use the vector identity
\[ \vec{\nabla} \cdot (\vec{A} f(r)) = \vec{A} \cdot \nabla f(r) + f(r) \vec{\nabla} \cdot \vec{A} \]

Substituting \( \vec{A} = \vec{P} \) and \( f(r) = 1/r \),
\[ \vec{\nabla} \cdot \left( \frac{\vec{P}}{r} \right) = \vec{P} \cdot \nabla \left( \frac{1}{r} \right) + \frac{1}{r} \vec{\nabla} \cdot \vec{P} \]

we get
\[ \phi = \frac{1}{4\pi \varepsilon_o} \int_{\text{vol}} \vec{\nabla} \cdot \left( \frac{\vec{P}}{r} \right) d\tau - \frac{1}{4\pi \varepsilon_o} \int_{\text{vol}} \frac{1}{r} \vec{\nabla} \cdot \vec{P} d\tau \]

The first integral can be converted to a surface integral using the divergence theorem giving,
\[ \phi = \frac{1}{4\pi \varepsilon_o} \int_{\text{surface}} \frac{\vec{P}}{r} \cdot d\vec{S} - \frac{1}{4\pi \varepsilon_o} \int_{\text{vol}} \frac{1}{r} \vec{\nabla} \cdot \vec{P} d\tau \]

The first term is the potential that one would expect for a surface charge density \( \sigma_b \) where
\[ \sigma_b = \vec{P} \cdot \hat{n} \]

where \( \hat{n} \) is the unit vector along outward normal to the surface. The second term is the potential due to a volume charge density \( \rho_b \) given by
\[ \rho_b = -\vec{\nabla} \cdot \vec{P} \]

The potential due to the dielectric is, therefore, given by
\[ \phi = \frac{1}{4\pi \varepsilon_o} \int_{\text{surface}} \frac{\sigma_b d\vec{S}}{r} + \frac{1}{4\pi \varepsilon_o} \int_{\text{vol}} \frac{\rho_b d\tau}{r} \]
and the electric field

\[ \vec{E} = -\nabla \phi \]

\[ = \frac{1}{4\pi \varepsilon_0} \int_{\text{surface}} \frac{\sigma_b \hat{\mathbf{r}}}{r^2} dS + \frac{1}{4\pi \varepsilon_0} \int_{\text{vol}} \frac{\rho_b \hat{\mathbf{r}}}{r^2} d\tau \]

### Gauss's Law in a Dielectric

We have seen that the effect of polarization of a dielectric is to produce bound charges of volume density \( \rho_b \) and surface density \( \sigma_b \), given by

\[ \rho_b = -\nabla \cdot \vec{P} \]

\[ \sigma_b = \vec{P} \cdot \hat{n} \]

The total electric field of a system which includes dielectrics is due to these polarization charge densities and other charges which may be present in the system. The latter are denoted as free charges to distinguish them from charges attributable to polarization effect. For instance, the valence charges in a metal or charges of ions embedded in a dielectric are considered as free charges. The total charge density of a medium is a sum of free and bound charges

\[ \rho = \rho_f + \rho_b \]

We can now formulate Gauss's law in the presence of a dielectric. Gauss's Law takes the form

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f + \rho_b}{\varepsilon_0} \]

Substituting \( \rho_b = -\nabla \cdot \vec{P} \), we get

\[ \nabla \cdot (\varepsilon \vec{E} + \vec{P}) = \rho_f \]

The electric displacement vector \( \vec{D} \) is defined by

\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \]

which has the same dimension as that of \( \vec{P} \). The equation satisfied by \( \vec{D} \) is thus,

\[ \nabla \cdot \vec{D} = \rho_f \]

which is the differential form of Gauss's law for a dielectric medium.

Integrating over the dielectric volume,

\[ \int_{\text{volume}} \nabla \cdot \vec{D} d\tau = \int_{\text{volume}} \rho_f d\tau = Q_f \]
where $Q_f$ is the free charge *enclosed* in the volume. The volume integral can be converted to a surface integral using the divergence theorem, which gives

$$\int_{\text{surface}} \vec{D} \cdot d\vec{S} = Q_f$$

Thus the flux over the vector $\vec{D}$ over a closed surface is equal to the free charged enclosed by the surface. The above formulations of Gauss’s law for dielectric medium are useful because they refer to only free charges for which we may have prior knowledge.

**Constitutive Relation**

Electric displacement vector $\vec{D}$ helps us to calculate fields in the presence of a dielectric. This is possible only if a relationship between $\vec{E}$ and $\vec{D}$ is known. For a weak to moderate field strength, the electric polarization $\vec{P}$ is found to be directly proportional to the external electric field $\vec{E}$. We define *Electric Susceptibility* $\chi$ through

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

so that

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \kappa \vec{E}$$

where $\kappa \equiv \varepsilon_r = 1 + \chi$ is called the *relative permittivity* or the dielectric constant and $\varepsilon$ is the permittivity of the medium. Using differential form of Gauss's law for $\vec{D}$, we get

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon} \nabla \cdot \vec{D} = \frac{P_f}{\varepsilon}$$

Thus the electric field produced in the medium has the same form as that in free space, except that the field strength is reduced by a factor equal to the dielectric constant $\kappa$. 