

# Numerical Differentiation and Integration

## Introduction

Numerical differentiation/ integration is the process of computing the value of the derivative of a function, whose analytical expression is not available, but is specified through a set of values at

certain tabular points  $x_0, x_1, \dots, x_n$ . In such cases, we first determine an interpolating polynomial approximating the function (either on the whole interval or in sub-intervals) and then differentiate/integrate this polynomial to approximately compute the value of the derivative at the given point.

## Numerical Differentiation

In the case of differentiation, we first write the interpolating formula on the interval  $(x_0, x_n)$  and then differentiate the polynomial term by term to get an approximated polynomial to the derivative of the function. When the tabular points are equidistant, one uses the Newton's Forward/ Backward Formula or Sterling's Formula; otherwise Lagrange's formula is used. Newton's Forward/ Backward formula is used depending upon the location of the point at which the derivative is to be computed. In case the given point is near the midpoint of the interval, Sterling's formula can be used. We illustrate the process by taking (i) Newton's Forward formula, and (ii) Sterling's formula.

Recall, that the Newton's forward interpolating polynomial is given by

$$f(x) = f(x_0 + hu) \approx y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!} (u(u-1)) + \dots + \frac{\Delta^k y_0}{k!} \{u(u-1) \dots (u-k) + \dots + \frac{\Delta^n y_0}{n!} \{u(u-1) \dots (u-n+1)\}. \quad (13.2.1)$$

Differentiating (13.2.1), we get the approximate value of the first derivative at  $x$  as

$$\frac{df}{dx} = \frac{1}{h} \frac{df}{du} \approx \frac{1}{h} \left[ \Delta y_0 + \frac{\Delta^2 y_0}{2!} (2u-1) + \frac{\Delta^3 y_0}{3!} (3u^2 - 6u + 2) + \dots \right]$$

$$+ \frac{\Delta^n y_0}{n!} \left( nu^{n-1} - \frac{n(n-1)^2}{2} u^{n-2} + \dots + (-1)^{(n-1)} (n-1)! \right) \Bigg] \quad (13.2.2)$$

$$u = \frac{x - x_0}{h},$$

where,

Thus, an approximation to the value of first derivative at  $x = x_0$  i.e.  $u = 0$  is obtained as :

$$\left. \frac{df}{dx} \right|_{x=x_0} = \frac{1}{h} \left[ \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \dots + (-1)^{(n-1)} \frac{\Delta^n y_0}{n} \right] \quad (13.2.3)$$

Similarly, using Stirling's formula:

$$\begin{aligned} f(x_0^* + hu) &\approx y_0^* + u \frac{\Delta y_{-1}^* + \Delta y_0^*}{2} + u^2 \frac{\Delta^2 y_{-1}^*}{2!} + \frac{u(u^2 - 1)}{2} \frac{\Delta^3 y_{-2}^* + \Delta^3 y_{-1}^*}{3!} \\ &\quad + u^2(u^2 - 1) \frac{\Delta^4 y_{-2}^*}{4!} + \frac{u(u^2 - 1)(u^2 - 2^2)}{2} \frac{\Delta^5 y_{-3}^* + \Delta^5 y_{-2}^*}{5!} + \dots \end{aligned} \quad (13.2.4)$$

Therefore,

$$\begin{aligned} \frac{df}{dx} &= \frac{1}{h} \frac{df}{du} \approx \frac{1}{h} \left[ \frac{\Delta y_{-1}^* + \Delta y_0^*}{2} + u \Delta^2 y_{-1}^* + \frac{(3u^2 - 1)}{2} \times \frac{(\Delta^3 y_{-2}^* + \Delta^3 y_{-1}^*)}{3!} \right. \\ &\quad \left. + 2u(2u^2 - 1) \frac{\Delta^4 y_{-2}^*}{4!} + \frac{(5u^4 - 15u^2 + 4)(\Delta^5 y_{-3}^* + \Delta^5 y_{-2}^*)}{2 \times 5!} + \dots \right] \end{aligned} \quad (13.2.5)$$

Thus, the derivative at  $x = x_0^*$  is obtained as:

$$\left. \frac{df}{dx} \right|_{x=x_0^*} = \frac{1}{h} \left[ \frac{\Delta y_{-1}^* + \Delta y_0^*}{2} - \frac{(1)}{2} \times \frac{(\Delta^3 y_{-2}^* + \Delta^3 y_{-1}^*)}{3!} + \frac{4 \times (\Delta^5 y_{-3}^* + \Delta^5 y_{-2}^*)}{2 \times 5!} + \dots \right] \quad (13.2.6)$$

**Remark 13.2.1** Numerical differentiation using Stirling's formula is found to be more accurate than that with the Newton's difference formulae. Also it is more convenient to use.

Now higher derivatives can be found by successively differentiating the interpolating

polynomials. Thus e.g. using (13.2.2), we get the second derivative at  $x = x_0$  as

$$\left. \frac{d^2 f}{dx^2} \right|_{x=x_0} = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{2 \times 11 \times \Delta^4 y_0}{4!} - \dots \right].$$

**EXAMPLE 13.2.2** Compute from following table the value of the derivative of  $y = f(x)$  at  $x = 1.7489$ :

$x$	1.73	1.74	1.75	1.76	1.77
$y$	1.772844100	1.155204006	1.737739435	1.720448638	1.703329888

**Solution:** We note here that  $x_0 = 1.75, h = 0.01, u = (1.7489 - 1.75)/0.01 = -0.11$ , so  $\Delta y_0 = -0.0017290797, \Delta^2 y_0 = .0000172047, \Delta^3 y_0 = -0.0000001712$ , and  $\Delta y_{-1} = -0.0017464571, \Delta^2 y_{-1} = .0000173774, \Delta^3 y_{-1} = -0.0000001727$ ,  $\Delta^3 y_{-2} = -0.0000001749, \Delta^4 y_{-2} = -0.0000000022$

Thus,  $f'(1.7489)$  is obtained as:

(i) Using Newton's Forward difference formula,

$$f'(1.4978) \approx \frac{1}{0.01} \left[ -0.0017290797 + (2 \times -0.11 - 1) \times \frac{0.0000172047}{2} + (3 \times (-0.11)^2 - 6 \times -0.11 + 2) \times \frac{-0.0000001712}{3!} \right] = -0.173965150143.$$

(ii) Using Stirling's formula, we get:

$$\begin{aligned} f'(1.4978) &\approx \frac{1}{.01} \left[ \frac{(-0.0017464571) + (-0.0017290797)}{2} + (-0.11) \times .0000173774 \right. \\ &\quad + \frac{(3 \times (-0.11)^2 - 1)}{2} \frac{((-0.0000001749) + (-0.0000001727))}{3!} \\ &\quad \left. + 2 \times (-0.11) \times (2(-0.11)^2 - 1) \times \frac{(-.0000000022)}{4!} \right] \\ &= -0.17396520185 \end{aligned}$$

$$f(x) = e^{-x}$$

It may be pointed out here that the above table is for  $f(x) = e^{-x}$ , whose derivative has the value -0.1739652000 at  $x = 1.7489$ .

**EXAMPLE 13.2.3** Using only the first term in the formula (13.2.6) show that

$$f'(x_0^*) \approx \frac{y_1^* - y_{-1}^*}{2h}.$$

Hence compute from following table the value of the derivative of  $y = e^x$  at  $x = 1.15$ :

$x$	1.05	1.15	1.25
$e^x$	2.8577	3.1582	3.4903

**Solution:** Truncating the formula (13.2.6) after the first term, we get:

$$\begin{aligned} f'(x_0^*) &\approx \frac{1}{h} \left[ \frac{\Delta y_{-1}^* + \Delta y_0^*}{2} \right] \\ &= \frac{(y_0^* - y_{-1}^*) + (y_1^* - y_0^*)}{2h} \\ &= \frac{y_1^* - y_{-1}^*}{2h}. \end{aligned}$$

Now from the given table, taking  $x_0^* = 1.15$ , we have

$$f'(1.15) \approx \frac{3.4903 - 2.8577}{2 \times 0.1} = 3.1630.$$

$$3.1630 - 3.1582 = 0.0048.$$

Note the error between the computed value and the true value is

**EXERCISE 13.2.4** Retaining only the first two terms in the formula (13.2.3), show that

$$f'(x_0) \approx \frac{-3y_0 + 4y_1 - y_2}{2h}.$$

Hence compute the derivative of  $y = e^x$  at  $x = 1.15$  from the following table:

$x$	1.15	1.20	1.25
$e^x$	3.1582	3.3201	3.4903

Also compare your result with the computed value in the example (13.2.3).

**EXERCISE 13.2.5** Retaining only the first two terms in the formula (13.2.6), show that

$$f'(x_0^*) \approx \frac{y_{-2}^* - 8y_{-1}^* + 8y_1^* - y_2^*}{12h}.$$

Hence compute from following table the value of the derivative of  $y = e^x$  at  $x = 1.15$ :

$x$	1.05	1.10	1.15	1.20	1.25
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$$e^x \quad 2.8577 \quad 3.0042 \quad 3.1582 \quad 3.3201 \quad 3.4903$$

**EXERCISE 13.2.6** Following table gives the values of  $y = f(x)$  at the tabular points  $x = 0 + 0.05 \times k \quad k = 0, 1, 2, 3, 4, 5$ .

$$x \quad 0.00 \quad 0.05 \quad 0.10 \quad 0.15 \quad 0.20 \quad 0.25$$

$$y \quad 0.00000 \quad 0.10017 \quad 0.20134 \quad 0.30452 \quad 0.41075 \quad 0.52110$$

Compute (i) the derivatives  $y'$  and  $y''$  at  $x = 0.0$  by using the formula (13.2.2). (ii) the second derivative  $y''$  at  $x = 0.1$  by using the formula (13.2.6).

Similarly, if we have tabular points which are not equidistant, one can use Lagrange's interpolating polynomial, which is differentiated to get an estimate of first derivative. We shall

see the result for four tabular points and then give the general formula. Let  $x_0, x_1, x_2, x_3$  be the tabular points, then the corresponding Lagrange's formula gives us:

$$\begin{aligned} f(x) \approx & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \end{aligned}$$

Differentiation of the above interpolating polynomial gives:

$$\begin{aligned} \frac{df(x)}{dx} \approx & \frac{(x-x_2)(x-x_3) + (x-x_1)(x-x_2) + (x-x_1)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) \\ & + \frac{(x-x_2)(x-x_3) + (x-x_0)(x-x_2) + (x-x_0)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ & + \frac{(x-x_1)(x-x_2) + (x-x_0)(x-x_1) + (x-x_0)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) \\ & + \frac{(x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\ = & \left( \prod_{r=0}^3 (x-x_r) \right) \left[ \sum_{i=0}^3 \frac{f(x_i)}{(x-x_i) \prod_{j=0, j \neq i}^3 (x_i-x_j)} \left( \sum_{k=0, k \neq i}^3 \frac{1}{(x-x_k)} \right) \right] \quad (13.2.7) \end{aligned}$$

In particular, the value of the derivative at  $x = x_0$  is given by

$$\left. \frac{df}{dx} \right|_{x=x_0} \approx \left[ \frac{1}{(x_0 - x_1)} + \frac{1}{(x_0 - x_2)} + \frac{1}{(x_0 - x_3)} \right] f(x_0) + \frac{(x_0 - x_2)(x_0 - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) \\ + \frac{(x_0 - x_1)(x_0 - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x_0 - x_1)(x_0 - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3).$$

Now, generalizing Equation (13.2.7) for  $n + 1$  tabular points  $x_0, x_1, \dots, x_n$  we get:

$$\frac{df}{dx} = \prod_{r=0}^n (x - x_r) \left[ \sum_{i=0}^n \frac{f(x_i)}{(x - x_i) \prod_{j=0, j \neq i}^n (x_i - x_j)} \left( \sum_{k=0, k \neq i}^n \frac{1}{(x - x_k)} \right) \right].$$

**EXAMPLE 13.2.7** Compute from following table the value of the derivative of  $y = f(x)$  at  $x = 0.6$ :

$x$	0.4	0.6	0.7
$y$	3.3836494	4.2442376	4.7275054

**Solution:** The given tabular points are not equidistant, so we use Lagrange's interpolating polynomial with three points:  $x_0 = 0.4, x_1 = 0.6, x_2 = 0.7$ .

Now differentiating this polynomial the derivative of the function at  $x = x_1$  is obtained in the following form:

$$\left. \frac{df}{dx} \right|_{x=x_1} \approx \frac{(x_1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \left[ \frac{1}{(x_1 - x_2)} + \frac{1}{(x_1 - x_0)} \right] f(x_1) + \frac{(x_1 - x_0)}{(x_2 - x_0)(x_2 - x_1)} f(x_2).$$

**Note:** The reader is advised to derive the above expression.

Now, using the values from the table, we get:

$$\begin{aligned}
 \left. \frac{df}{dx} \right|_{x=0.6} &\approx \frac{(0.6 - 0.7)}{(0.4 - 0.6)(0.4 - 0.7)} \times 3.3836494 + \left[ \frac{1}{(0.6 - 0.7)} + \frac{1}{(0.6 - 0.4)} \right] \times 4.2442376 \\
 &\quad + \frac{(0.6 - 0.4)}{(0.7 - 0.4)(0.7 - 0.6)} \times 4.7225054 \\
 &= -5.63941567 - 21.221188 + 31.48336933 = 4.6227656.
 \end{aligned}$$

*For the sake of comparison, it may be pointed out here that the above table is for the function  $f(x) = 2e^x + x$ , and the value of its derivative at  $x = 0.6$  is 4.6442376.*

**EXERCISE 13.2.8** *For the function, whose tabular values are given in the above example(13.2.8), compute the value of its derivative at  $x = 0.5$ .*

**Remark 13.2.9** *It may be remarked here that the numerical differentiation for higher derivatives does not give very accurate results and so is not much preferred.*

# Numerical Integration

$$\int_a^b f(x)dx,$$

Numerical Integration is the process of computing the value of a definite integral,

$$y = f(x)$$

when the values of the integrand function,  $y = f(x)$  are given at some tabular points. As in the case of Numerical differentiation, here also the integrand is first replaced with an interpolating polynomial, and then the integrating polynomial is integrated to compute the value of the definite integral. This gives us 'quadrature formula' for numerical integration. In the case of equidistant tabular points, either the Newton's formulae or Stirling's formula are used. Otherwise, one uses Lagrange's formula for the interpolating polynomial. We shall consider below the case of

$$x_0, x_1, \dots, x_n.$$

equidistant points:

## A General Quadrature Formula

Let  $f(x_k) = y_k$  be the nodal value at the tabular point  $x_k$  for  $k = 0, 1, \dots, n$ , where  $x_0 = a$  and  $x_n = x_0 + nh = b$ .

Now, a general quadrature formula is obtained by replacing the integrand by Newton's forward difference interpolating polynomial. Thus, we get,

$$\int_a^b f(x)dx = \int_a^b \left[ y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \frac{\Delta^4 y_0}{4!h^4}(x - x_0)(x - x_1)(x - x_2)(x - x_3) + \dots \right] dx$$

This on using the transformation  $x = x_0 + hu$  gives:

$$\int_a^b f(x)dx = h \int_0^n \left[ y_0 + u\Delta y_0 + \frac{\Delta^2 y_0}{2!}u(u-1) + \frac{\Delta^3 y_0}{3!}u(u-1)(u-2) + \dots \right] du$$



$$+ \frac{\Delta^4 y_0}{4!} u(u-1)(u-2)(u-3) + \dots \Big] du$$

which on term by term integration gives,

$$\begin{aligned} \int_a^b f(x) dx &= h \left[ n y_0 + \frac{n^2}{2} \Delta y_0 + \frac{\Delta^2 y_0}{2!} \left( \frac{n^3}{3} - \frac{n^2}{2} \right) + \frac{\Delta^3 y_0}{3!} \left( \frac{n^4}{4} - n^3 + n^2 \right) \right. \\ &\quad \left. + \frac{\Delta^4 y_0}{4!} \left( \frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2 \right) + \dots \right] \end{aligned} \quad (13.3.1)$$

For  $n = 1$ ,  
i.e., when linear interpolating polynomial is used then, we have

$$\int_a^b f(x) dx = h \left[ y_0 + \frac{\Delta y_0}{2} \right] = \frac{h}{2} [y_0 + y_1]. \quad (13.3.2)$$

Similarly, using interpolating polynomial of degree 2 (i.e.  $n = 2$ ), we obtain,

$$\begin{aligned} \int_a^b f(x) dx &= h \left[ 2y_0 + 2\Delta y_0 + \left( \frac{8}{3} - \frac{4}{2} \right) \frac{\Delta^2 y_0}{2} \right] \\ &= 2h \left[ y_0 + (y_1 - y_0) + \frac{1}{3} \times \frac{y_2 - 2y_1 + y_0}{2} \right] = \frac{h}{3} [y_0 + 4y_1 + y_2]. \end{aligned} \quad (13.3.3)$$

In the above we have replaced the integrand by an interpolating polynomial over the whole interval

$[a, b]$

and then integrated it term by term. However, this process is not very useful. More useful

Numerical integral formulae are obtained by dividing the interval  $[a, b]$  in  $n$  sub-intervals  $[x_k, x_{k+1}]$ , where,  $x_k = x_0 + kh$  for  $k = 0, 1, \dots, n$  with  $x_0 = a, x_n = x_0 + nh = b$ .

### Trapezoidal Rule

Here, the integral is computed on each of the sub-intervals by using linear interpolating formula, *i.e.* for  $n = 1$  and then summing them up to obtain the desired integral.

Note that

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \dots + \int_{x_{k+1}}^{x_k} f(x)dx + \dots + \int_{x_n}^{x_{n-1}} f(x)dx$$

Now using the formula (13.3.2) for  $n = 1$  on the interval  $[x_k, x_{k+1}]$ , we get,

$$\int_{x_k}^{x_{k+1}} f(x)dx = \frac{h}{2} [y_k + y_{k+1}] .$$

Thus, we have,

$$\int_a^b f(x)dx = \frac{h}{2} [y_0 + y_1] + \frac{h}{2} [y_1 + y_2] + \dots + \frac{h}{2} [y_k + y_{k+1}] + \dots + \frac{h}{2} [y_{n-2} + y_{n-1}] + \frac{h}{2} [y_{n-1} + y_n]$$

*i.e.*

$$\begin{aligned} \int_a^b f(x)dx &= \frac{h}{2} [y_0 + 2y_1 + 2y_2 + \dots + 2y_k + \dots + 2y_{n-1} + y_n] \\ &= h \left[ \frac{y_0 + y_n}{2} + \sum_{i=1}^{n-1} y_i \right] . \end{aligned} \quad (13.3.4)$$

This is called TRAPEZOIDAL RULE. It is a simple quadrature formula, but is not very accurate.

**Remark 13.3.1** An estimate for the error  $E_1$  in numerical integration using the Trapezoidal rule is given by

$$E_1 = -\frac{b-a}{12} \overline{\Delta^2 y},$$

where  $\overline{\Delta^2 y}$  is the average value of the second forward differences.

Recall that in the case of linear function, the second forward differences is zero, hence, the Trapezoidal rule gives exact value of the integral if the integrand is a linear function.

$$\int_0^1 e^{x^2} dx,$$

**EXAMPLE 13.3.2** Using Trapezoidal rule compute the integral  $\int_0^1 e^{x^2} dx$  where the table for the values

of  $y = e^{x^2}$  is given below:

$x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y$	1.00000	1.01005	1.04081	1.09417	1.17351	1.28402	1.43332	1.63231	1.89648	2.2479	2.71828

$$h = 0.1, n = 10,$$

**Solution: Here,**

$$\frac{y_0 + y_{10}}{2} = \frac{1.0 + 2.71828}{2} = 1.85914,$$

and

$$\sum_{i=1}^9 y_i = 12.81257.$$

**Thus,**

$$\int_0^1 e^{x^2} dx = 0.1 \times [1.85914 + 12.81257] = 1.467171$$

## Simpson's Rule

If we are given odd number of tabular points, i.e.  $n$  is even, then we can divide the given integral of

integration in even number of sub-intervals  $[x_{2k}, x_{2k+2}]$ . Note that for each of these sub-intervals, we

have the three tabular points  $x_{2k}, x_{2k+1}, x_{2k+2}$  and so the integrand is replaced with a quadratic interpolating polynomial. Thus using the formula (13.3.3), we get,

$$\int_{x_{2k}}^{x_{2k+2}} f(x) dx = \frac{h}{3} [y_{2k} + 4y_{2k+1} + y_{2k+2}].$$

In view of this, we have

$$\begin{aligned} \int_a^b f(x) dx &= \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \cdots + \int_{x_{2k}}^{x_{2k+2}} f(x) dx + \cdots + \int_{x_{n-2}}^{x_n} f(x) dx \\ &= \frac{h}{3} [(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)] \\ &= \frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n], \end{aligned}$$

which gives the second quadrature formula as follows:

$$\begin{aligned} \int_a^b f(x) dx &= \frac{h}{3} [(y_0 + y_n) + 4 \times (y_1 + y_3 + \cdots + y_{2k+1} + \cdots + y_{n-1}) \\ &\quad + 2 \times (y_2 + y_4 + \cdots + y_{2k} + \cdots + y_{n-2})] \\ &= \frac{h}{3} \left[ (y_0 + y_n) + 4 \times \left( \sum_{i=1, i-\text{odd}}^{n-1} y_i \right) + 2 \times \left( \sum_{i=2, i-\text{even}}^{n-2} y_i \right) \right] \quad (13.3.5) \end{aligned}$$

This is known as SIMPSON'S RULE.

**Remark 13.3.3** An estimate for the error  $E_2$  in numerical integration using the Simpson's rule is given by

$$E_2 = -\frac{b-a}{180} \overline{\Delta^4 y}, \quad (13.3.6)$$

where  $\overline{\Delta^4 y}$  is the average value of the forth forward differences.

$$y = e^{x^2}$$

**EXAMPLE 13.3.4** Using the table for the values of  $y = e^{x^2}$  as is given in Example 13.3.2, compute the

$$\int_0^1 e^{x^2} dx,$$

integral by Simpson's rule. Also estimate the error in its calculation and compare it with the error using Trapezoidal rule.

**Solution:** Here,  $h = 0.1, n = 10$ , thus we have odd number of nodal points. Further,

$$y_0 + y_{10} = 1.0 + 2.71828 = 3.71828, \quad \sum_{i=1, i-\text{odd}}^9 y_i = y_1 + y_3 + y_5 + y_7 + y_9 = 7.26845,$$

and

$$\sum_{i=2, i-\text{even}}^8 y_i = y_2 + y_4 + y_6 + y_8 = 5.54412.$$

Thus,

$$\int_0^1 e^{x^2} dx = \frac{0.1}{3} \times [3.71828 + 4 \times 7.268361 + 2 \times 5.54412] = 1.46267733$$

To find the error estimates, we consider the forward difference table, which is given below:

$x_i$	$y_i$	$\Delta y_i$	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$		
0.0	1.00000	0.01005	0.02071	0.00189	0.00149		
0.1	1.01005	0.03076	0.02260	0.00338	0.00171		
0.2	1.04081	0.05336	0.02598	0.00519	0.00243		
0.3	1.09417	0.07934	0.03117	0.00762	0.00320		
0.4	1.17351	0.11051	0.3879	0.01090	0.00459		
0.5	1.28402	0.14930	0.04969	0.01549	0.00658		
0.6	1.43332	0.19899	0.06518	0.02207	0.00964		
0.7	1.63231	0.26417	0.08725	0.03171			
0.8	1.89648	0.35142	0.11896				
0.9	2.24790	0.47038					
1.0	2.71828						

Thus, error due to Trapezoidal rule is,

$$\begin{aligned}
 E_1 &= -\frac{1-b}{12} \Delta^2 y \\
 &= -\frac{1}{12} \times \frac{0.02071 + 0.02260 + 0.02598 + 0.03117 + 0.03879 + 0.04969 + 0.06518 + 0.0}{9} \\
 &= -0.004260463.
 \end{aligned}$$

Similarly, error due to Simpson's rule is,

$$\begin{aligned}
 E_2 &= -\frac{1-0}{180} \Delta^4 y \\
 &= -\frac{1}{180} \times \frac{0.00149 + 0.00171 + 0.00243 + 0.00328 + 0.00459 + 0.00658 + 0.00964}{7} \\
 &= -2.35873 \times 10^{-5}.
 \end{aligned}$$

It shows that the error in numerical integration is much less by using Simpson's rule.

EXAMPLE 13.3.5 Compute the integral  $\int_{0.05}^1 f(x) dx$ , where the table for the values of  $y = f(x)$  is given below:

$x$	0.05	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y$	0.0785	0.1564	0.2334	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1.0000

**Solution:** Note that here the points are not given to be equidistant, so as such we can not use any of the above two formulae. However, we notice that the tabular points  $0.05, 0.10, 0.15$  and  $0.20$  are equidistant and so are the tabular points  $0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$  and  $1.0$ .

Now we can divide the interval in two subinterval:  $[0.05, 0.2]$  and  $[0.2, 1.0]$ ; thus,

$$\int_{0.05}^1 f(x) dx = \int_{0.05}^{0.2} f(x) dx + \int_{0.2}^1 f(x) dx$$

. The integrals then can be evaluated in each interval. We observe that the second set has odd number of points. Thus, the first integral is evaluated by using Trapezoidal rule and the second one by Simpson's rule (of course, one could have used Trapezoidal rule in both the subintervals).

For the first integral  $h = 0.05$  and for the second one  $h = 0.1$ . Thus,

$$\int_{0.05}^{0.2} f(x)dx = 0.05 \times \left[ \frac{0.0785 + 0.3090}{2} + 0.1564 + 0.2334 \right] = 0.0291775,$$

$$\begin{aligned} \text{and } \int_{0.2}^{1.0} f(x)dx &= \frac{0.1}{3} \times \left[ (0.3090 + 1.0000) + 4 \times (0.4540 + 0.7071 + 0.8910 + 0.9877) \right. \\ &\quad \left. + 2 \times (0.5878 + 0.8090 + 0.9511) \right] \\ &= 0.6054667, \end{aligned}$$

which gives,

$$\int_{0.05}^1 f(x)dx = 0.0291775 + 0.6054667 = 0.6346442$$

$$f(x) = \sin(\pi x/2)$$

*It may be mentioned here that in the above integral, and that the value of the integral is 0.6346526. It will be interesting for the reader to compute the two integrals using Trapezoidal rule and compare the values.*

#### EXERCISE 13.3.6

$$\int_a^b f(x)dx,$$

- Using Trapezoidal rule, compute the integral where the table for the values of  $y = f(x)$  is given below. Also find an error estimate for the computed value.

1.

$x$	a=1	2	3	4	5	6	7	8	9	b=10
$y$	0.0953	0.1823	0.2623	0.3364	0.4054	0.4700	0.5306	0.5877	0.6418	0.6931
	1	2	6	7	6	0	3	9	5	4



3.

$x$    a=1.50   1.55   1.60   1.65   1.70   1.75   b=1.80

$y$    0.40546   0.43825   0.47000   0.5077   0.53063   0.55962   0.58779

5.

$x$    a = 1.0   1.5   2.0   2.5   3.0   b = 3.5

$y$    1.1752   2.1293   3.6269   6.0502   10.0179   16.5426

$$\int_a^b f(x) dx.$$

2. Using Simpson's rule, compute the integral  $\int_a^b f(x) dx$ . Also get an error estimate of the computed integral.

1. Use the table given in Exercise 13.3.6.1b.
- 2.

$x$    a = 0.5   1.0   1.5   2.0   2.5   3.0   b = 3.5

$y$    0.493   0.946   1.325   1.605   1.778   1.849   1.833

$$\int_0^{1.5} f(x) dx$$

3. Compute the integral  $\int_0^{1.5} f(x) dx$ , where the table for the values of  $y = f(x)$  is given below:

$x$    0.0   0.5   0.7   0.9   1.1   1.2   1.3   1.4   1.5

$y$    0.00   0.39   0.77   1.27   1.90   2.26   2.65   3.07   3.53