Module 1: Signals & System

Lecture 6: Basic Signals in Detail

Basic Signals in detail

We now introduce formally some of the basic signals namely

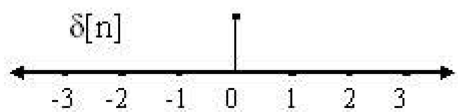
1) The Unit Impulse function.

2) The Unit Step function

These signals are of considerable importance in signals and systems analysis. Later in the course we will see these signals as the building blocks for representation of other signals. We will cover both signals in continuous and discrete time. However, these concepts are easily comprehended in Discrete Time domain, so we begin with Discrete Time Unit Impulse and Unit Step function.

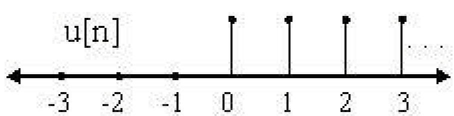
The Discrete Time Unit Impulse Function: This is the simplest discrete time signal and is defined as

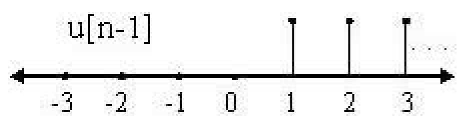
$$\begin{aligned}\delta[n] &= 0 \quad \forall n \neq 0 \\ &= 1 \quad for\ n = 0\end{aligned}$$



The Discrete Time Unit Step Function u[n]: It is defined as

$$\begin{aligned}u[n] &= 0 \quad \forall\ n < 0 \\ &= 1 \quad \forall\ n \geq 0\end{aligned}$$





Unit step in terms of unit impulse function

Having studied the basic signal operations namely **Time Shifting, Time Scaling** and **Time Inversion** it is easy to see that

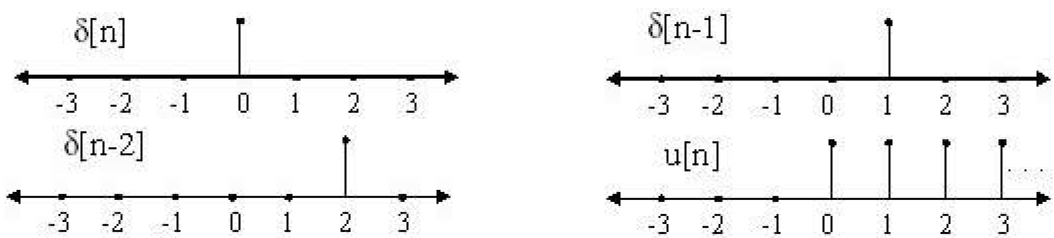
$$\delta[n] = u[n] - u[n-1]$$

Similarly,

$$\delta[n-1] = u[n-1] - u[n-2]$$
$$\delta[n-2] = u[n-2] - u[n-3]$$
$$\delta[n-3] = u[n-3] - u[n-4]$$
$$\delta[n-4] = u[n-4] - u[n-5]$$

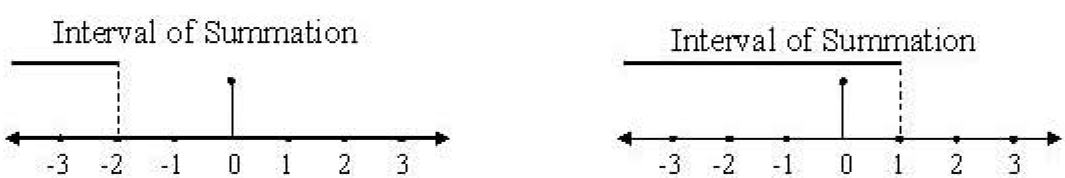
Summing over we get $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

Looking directly at the Unit Step Function we observe that it can be constructed as a sum of shifted Unit Impulse Functions



The unit function can also be expressed as a running sum of the Unit Impulse Function

$$u[n] = \sum_{k=-\infty}^{n} \delta(k)$$



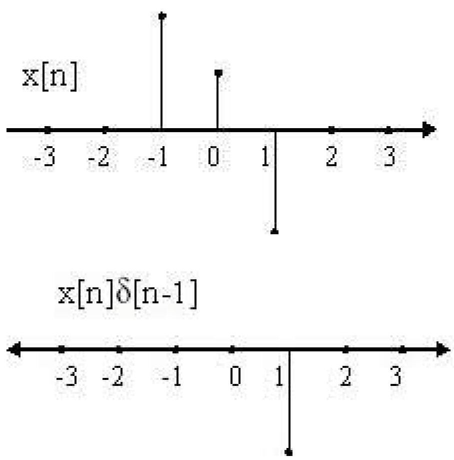
We see that the running sum is **0** for **n < 0** and equal to **1** for **n >= 0** thus defining the Unit Step Function **u[n]**.

Sifting property

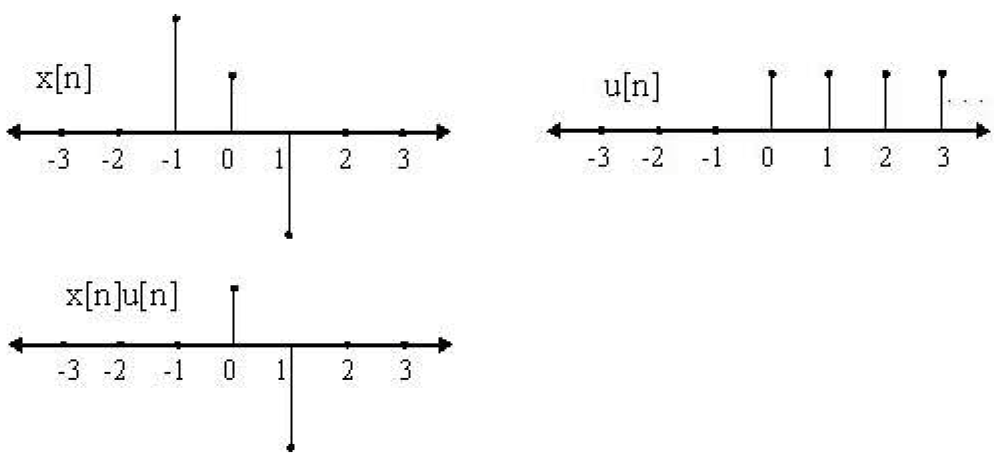
Consider the product $x[n]\delta[n]$. The delta function is non zero only at the origin so it follows the signal is the same as $x[0]\delta[n]$.

More generally $x[n]\delta[n-k] \quad = \quad x[k]\delta[n-k]$

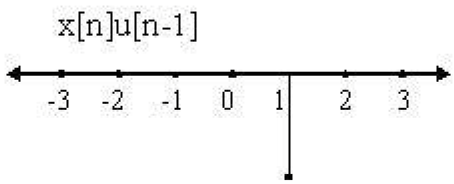
It is important to understand the above expression. It means the product of a given signal **x[n]** with the shifted Unit Impulse Function is equal to the time shifted Unit Impulse Function multiplied by **x[k]**. Thus the signal is *0* at time *not equal to k* and at time **k** the amplitude is **x[k]**. So we see that the unit impulse sequence can be used to obtain the value of the signal at any time k. This is called the Sampling Property of the Unit Impulse Function. This property will be used in the discussion of LTI systems. For example consider the product $x[n]\delta[n-1]$**. It gives** $x[1]\delta[n-1]$.



Likewise, the product **x[n] u[n]** i.e. the product of the signal **u[n]** with **x[n]** truncates the signal for **n < 0** since **u[n] = 0 for n <0**



Similarly, the product **x[n] u[n-1]** will truncate the signal for **n < 1**.

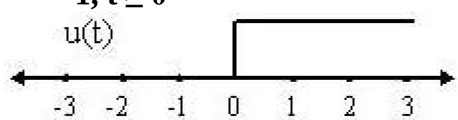


Now we move to the Continuous Time domain. We now introduce the **Continuous Time Unit Impulse Function and Unit Step Function**.

Continuous time unit step and unit impulse functions

The Continuous Time Unit Step Function: The definition is analogous to its Discrete Time counterpart i.e.

u(t) = 0, t < 0
= 1, t ≥ 0



The unit step function is discontinuous at the origin.

The Continuous Time Unit Impulse Function: The unit impulse function also known as the Dirac Delta Function, was first defined by Dirac as

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

In the strict mathematical sense the impulse function is a rather delicate concept. The Impulse function is not an ordinary function. An ordinary function is defined at all values of t. The impulse function is 0 everywhere except at t = 0 where it is undefined. This difficulty is resolved by defining the function as a GENERALIZED FUNCTION. **A generalized function is one which is defined by its effect on other functions instead of its value at every instant of time.**

Analogy from discrete domain

We will see that the impulse function is defined by its sampling property. We shall develop the theory by drawing analogy from the Discrete Time domain. Consider the equation

$$u[n] = \sum_{m=-\infty}^{\infty} \delta[m]$$

The discrete time unit step function is a running sum of the delta function. The continuous time unit impulse and unit step function are then related by

$$u(t) = \int_{-\infty}^{t} \delta(t)\, dt$$

The continuous time unit step function is a running integral of the delta function. It follows that the continuous time unit impulse can be thought of as the derivative of the continuous time unit step function.

$$\delta(t) = \frac{d\,u(t)}{dt}$$

Now here arises the difficulty. The unit Step function is not differentiable at the origin. We take a different approach. Consider the signal whose value increases from 0 to 1 in a short interval of time say delta. The function u(t) can be seen as the limit of the above signal as delta tends to 0. Given this definition of Unit Step function we look into its derivative. The unit impulse function can be regarded as a rectangular pulse with a width of Δ and height (1 / Δ). As Δ tends to 0 the function approaches the Unit Impulse function and its derivative becomes narrower and higher and eventually a pulse of infinitesimal width of infinite height. All throughout the area under the pulse is maintained at unity no matter the value of Δ . In effect the delta function has no duration but unit area. Graphically the function is denoted as spear like symbol at t = 0 and the "1" next to the arrow indicates the area of the impulse. After this discussion we have still not cleared the ambiguity regarding the value or the shape of the Unit Impulse Function at t = 0. We were only able to derive that the the effective duration of the pulse approaches zero while maintaining its area at unity. As we said earlier an Impulse Function is a Generalized Function and is defined by its effect on other functions and not by its value at every instant of time. Consider the product of an impulse function and a more well behaved continuous function. We will take the impulse function as the limiting case of a rectangular pulse of width Δ and height (1/ Δ) as earlier. As evident from the figure the product function is 0 everywhere except in the small interval. In this interval the value of x(t) can be assumed to be constant and equal to x(0). Thus the product function is equal to the function scaled by a value equal to x(0). Now as Δ tends to 0 the product tends to x(0) times the impulse function.

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\, dt = x(0) \int_{-\infty}^{\infty} \delta(t)\, dt$$

$$= x(0)$$

i.e. The area under the product of the signal and the unit impulse function is equal to the value of the signal at the point of impulse. This is called the Sampling Property of the Delta function and defines the impulse function in the generalized function approach. As in discrete time

$$x(t)\delta(t) = x(0)\delta(t)$$

Or more generally,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Also the product **x(t)u(t)** truncates the signal for **t < 0.**

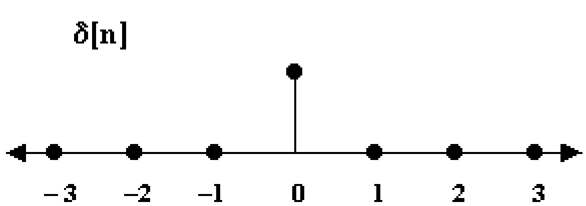
Lecture 7: Linear Shift Invariant Systems

Linear Shift-Invariant systems, called LSI systems for short, form a very important class of practical systems, and hence are of interest to us. They are also referred to as Linear Time-Invariant systems, in case the independent variable for the input and output signals is time. Remember that linearity means that is $y_1(t)$ and $y_2(t)$ are responses of the system to signals $x_1(t)$ and $x2(t)$ respectively, then the response to $ax_1(t) + bx_2(t)$ is $ay_1(t) + by_2(t)$.

Shift invariance implies that the response of the system to $x_1(t - t_0)$ is given by $y_1(t - t_0)$ for all values of t and t_0. Linear systems are of interest to us for primarily two reasons: first, several real-life systems can be well approximated by linear systems. Second, linear systems come with several properties which make their analysis simple. Similarly, shift-invariant systems allow us to use simpler math to analyse the system. As we proceed with our analysis, we will point out cases where some results (which are rather intuitive) are valid for only LSI systems.

The unit impulse (discrete time):

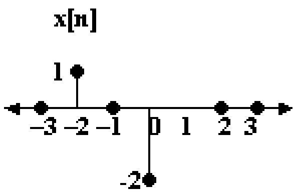
How do we go on with studying the responses of systems to various signals? It would be great if we can study the response of the system to one (or a few) signal(s) and predict the responses to all signals. It turns out that LSI systems can in fact be treated in such manner. The signal whose response we study is the unit impulse signal. If we know the response of the system to the unit impulse (called, for obvious reasons, the unit impulse response), then the system is completely characterized - we can find the response of the system to all possible inputs. This follows rather intuitively in discrete signals, so let us begin our analysis with discrete signals. In discrete signals, the unit impulse is a signal which has zero values everywhere except at one point, where its values is 1. Typically, this point is taken to be the origin (n=0).

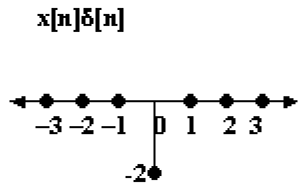


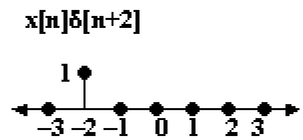
The unit impulse is denoted by the Greek letter delta δ . For example, the above impulses are denoted by $\delta[n]$ and $\delta[n-4]$ respectively.

Note: We are towards invoking shift invariance of the system here - we have shifted the signal $\delta[n]$ by 4 units.

We can thus use $\delta[n]$ to pick up a certain point from a discrete signal: suppose our signal x[n] is multiplied by $\delta[n-k]$ then the value of $x_1[n] = x[n]\delta[n-k]$ is zero at all point except **n=k**. At this point, the value of $\mathbf{x_1[k]}$ equals the value **x[k].**



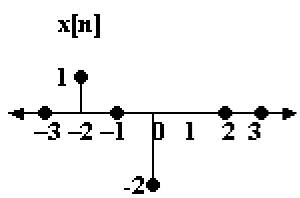


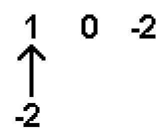


Now, we can express any discrete signal as a sum of several such terms:

$$\sum_{k=-\infty}^{+\infty} x[n].\delta[k]$$

This may seem redundant now, but later we shall find this notation useful when we take a look at convolutions etc. Here, we also want to introduce a convention for denoting discrete signals. For example, the signal x[n] and its representation are shown below:





The number below the arrow shows the starting point of the time sequence, and the numbers above are the values of the dependent variable at successive instants from then onwards. We may not use this too much on the web site, but this turns out to be a convenient notation on paper.

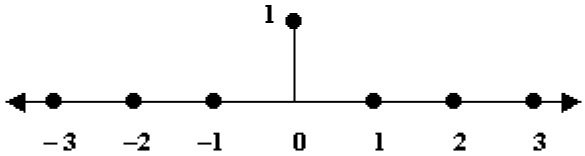
The unit impulse response:

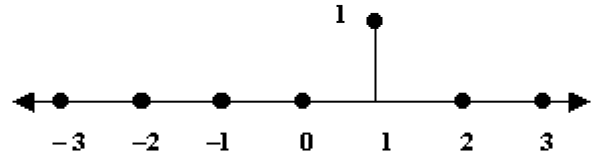
The response of a system to the unit impulse is of importance, for as we shall show below, it characterizes the LSI system completely. Let us consider the following system and calculate the unit step response to it: **y[n] = x[n] - 2x[n-1] + 3x[n-2]**. Now, we apply a unit step **x[n]=δ[n]** to the system and calculate the response :

$y[n] = x[n] - 2x[n-1] + 3x[n-2] \mid x[n] = \delta[n]$				
n	**x[n]**	**x[n-1]**	**x[n-2]**	**y[n]**
..., -1	0	0	0	**0**
0	1	0	0	**1**
1	0	1	0	**-2**
2	0	0	1	**3**
3, ...	0	0	0	**0**

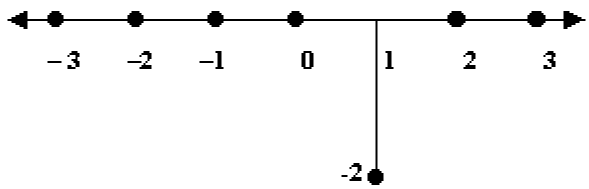
The graphical calculation and the response are as follows:

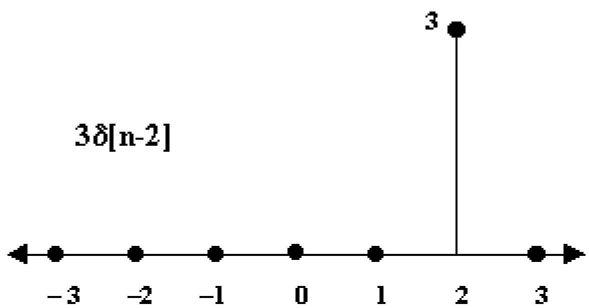
$\delta[n]$ $\delta[n-1]$

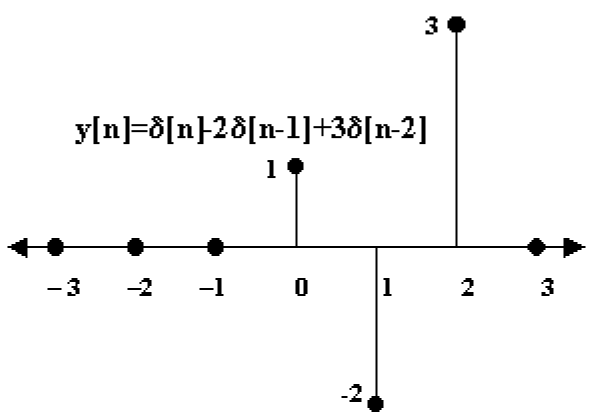




$-2\delta[n-1]$





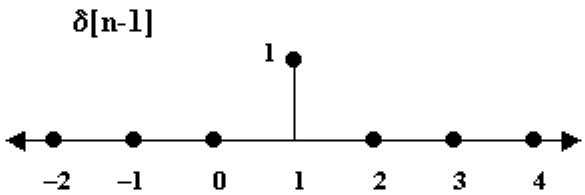


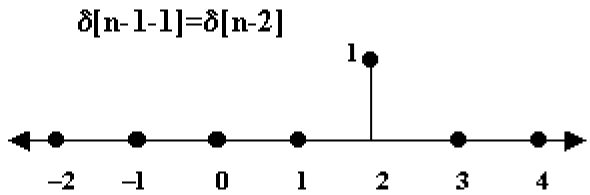
Arbitrary input signals:

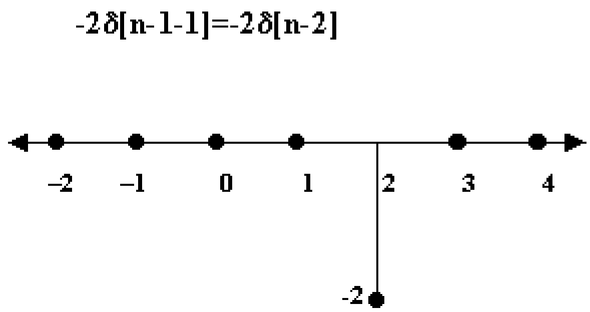
Now let us consider some other input, say **x[0]=1, x[1]=1** and **x=0** for n other than 0 and 1. What will be the response of the above LSI system to this input? We calculate the response in a table as below

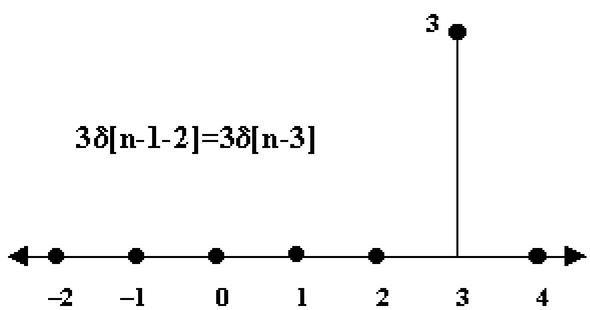
y[n] = x[n] - 2x[n-1] + 3x[n-2]		**x[n]=** $\delta[n]$ + $\delta[n-1]$	
n	**y_1[n] from** $\delta[n]$	**y_2[n] from -** $\delta[n-1]$	**y[n] = y_1[n] + y_2[n]**
..., -1	0	0	**0**
0	1	0	**1**
1	-2	1	**-1**
2	3	-2	**1**
3	0	3	**3**
4, ...	0	0	**0**

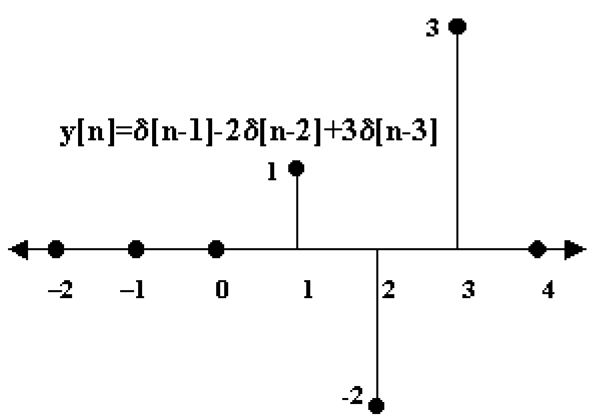
Ah! What we have actually done, is applied the additive (linear), homogenous (linear) and shift invariance properties of the system to get the output. First, we decomposed the input signal as a sum of known signals: first being the unit step $\delta[n]$. The second signal is derived from the unit step by shifting it by **1**. Thus, our input signal is as shown in the figure below. Then, we invoke the LSI properties of the system to get the responses to the individual signals: the first calculation is show above, while the calculation of response for $\delta[n-1]$ is shown below.



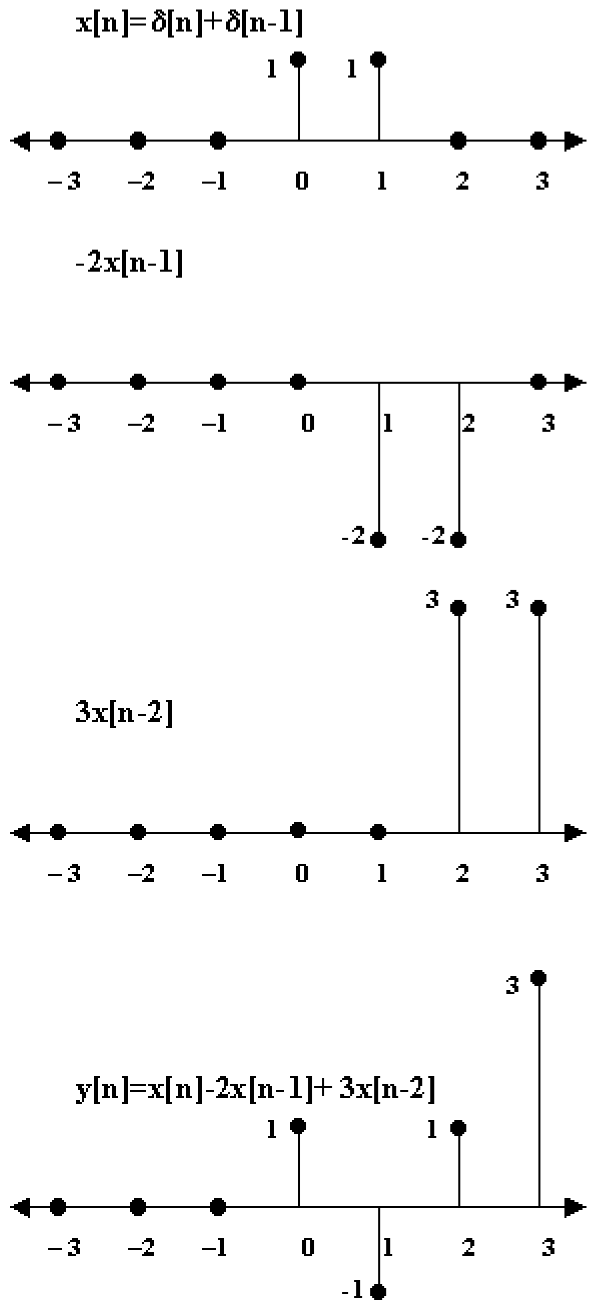








Finally, we add the two responses to get the response **y[n]** of the system to the input **x[n]**. The image below shows the final response with an alternative method of calculating it:



This brings us up to the concept of **convolutions**, covered in detail in a later section.

Lecture 8: Classification of Systems

Properties of discrete variable systems

We have classified systems into three classes - Continuous-time systems, Discrete-time systems and Hybrid systems. Now that we have introduced some system properties, let us see what properties are relevant to which classes of systems.

Let us first consider examples of different classes of systems.

Continuous-time systems Continuous-Continuous systems 1. Tree swaying in the wind: Wind - described by its speed, direction - is a continuous-time input. Movement of branches is continuous-time output signal.	**Discrete-time systems** Discrete-Discrete systems 1. Logic circuits: Discrete logic inputs are processed to give discrete logic outputs.
Hybrid systems Continuous-Discrete systems 1.Eye: sees continuous image, but sends a discrete map to the brain 2.Computer microphone: Sampler converts a continuous time signal into a discrete time signal.(Sampler forms an important system in today's digital world - we shall look at this in great detail later in the course)	**Hybrid systems** Discrete-Continuous systems 1.Brain : gets a discrete map from the eye, and completes a smooth, continuous picture 2. Computer speaker and sound card - a digital music output given by the computer is smoothed out and played as a continuous waveform.

Properties of systems

In early parts of this course, we shall concern ourselves with mainly the first two classes, viz. Continuous-time and Discrete-time systems, but later we shall also deal with Hybrid systems as well. So, we find it worthwhile here to take a look at what properties the systems of various classes **can** have:

Property	**Continuous input - Continuous output**	**Discrete input-Discrete output**	**Continuous-Discrete input/ Discrete-Continuous output**
Memory	Yes If input and output are of the same type	Yes If input and output are of the same type	No However, we can define a restricted version of memory if there is a correspondence in the input and output variables (e.g.: continuous and discrete time)
Causality	Yes If input and output are of the same type	Yes If input and output are of the same type	No A restricted version of causality can be defined: "If the inputs are same upto an instant corresponding to a discrete variable, then the outputs of a causal system are same
Shift invariance (Time invariance)	Yes If input and output are of the same type	Yes If input and output are of the same type	No We can define shift invariance in cases where the inputs are shifted by certain quanta corresponding to the spacing in discrete variables.
Stability	Yes	Yes	Yes
Linearity	Yes	Yes	Yes

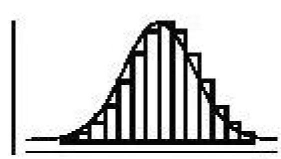
Note that this is a table of properties which the system **can** have; they are not necessary properties of a system. Hence, we can find a Continuous-time system that is stable (though there may be Continuous-time systems which are unstable), but it is impossible to apply the concept of memory to a discrete-continuous system without modifying the concept itself.

Lecture 9: Continuous LTI Systems

In this section our goal is to derive the response of a LTI system for any arbitrary continuous input **x(t)**. In complete analogy with the discussion on Discrete time analysis we begin by expressing **x(t)** in terms of impulses. In discrete time we represented a signal in terms of scaled and shifted unit impulses. In continuous time, however the unit impulse function is not an ordinary function (i.e. it is not defined at all points and we prefer to call the unit impulse function a "mathematical object"), it is a generalized function (it is defined by its effect on other signals) .

Recall the previous discussion on the development of the unit impulse function. It can be regarded as the idealization of a pulse of width Δ and height 1/ Δ .

One can arrive at an expression for an arbitrary input, say **x(t)** by scaling the height of the rectangular impulse by a factor such that it's value at t coincides with the value of **x(t)** at the mid-point of the width of the rectangular impulse. The entire function is hence divided into such rectangular impulses which give a close approximation to the actual function depending upon how small the interval is taken to be. For example let **x(t)** be a signal. It can be approximated as :



The given input **x(t)** is approximated with such narrow rectangular pulses, each scaled to the appropriate value of **x(t)** at the corresponding **t** (which lies at the midpoint of the base of width Δ . This is called the **staircase approximation** of **x(t)**. In the limit as the pulse-width (Δ) approaches zero, the rectangular pulse becomes finer in width and the function **x(t)** can be represented in terms of impulses by the following expression,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{k=\infty} x(k\Delta) \cdot \delta(t - k\Delta) \cdot \Delta$$

This summation is an approximation. As Δ approaches zero, the approximation increases in accuracy and when delta becomes infinitesimally small, this error becomes zero and the above summation is converted into the following integral expression.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

For example, take **x(t) = u(t)**

$$u(t) = \int_{-\infty}^{\infty} \mathbf{u}(\tau)\, \delta(t-\tau)\, d\tau = \int_{0}^{\infty} \mathbf{u}(\tau)\, \delta(t-\tau)\, d\tau$$

since **u(t) = 0** for **t < 0** and **u(t) = 1** for **t > 0**. In complete analogy with the development on sampling property of discrete unit impulse we have,

$$x(t)\delta(t-t_o) = x(t_o)\delta(t-t_o)$$

This is known as **Sifting Property** of the continuous time impulse. Note that the unit impulse puts unit area into zero width.

The Convolution Integral

We now want to find the response for an arbitrary continuous time signal as the superposition of scaled and shifted pulses just as we did for discrete time signal. For a continuous LSI system, let **h(t)** be the response to the unit impulse signal. Then,

$$\delta(t) \rightarrow h(t)$$

by shift invariance,

$$\delta(t-\lambda) \rightarrow h(t-\lambda)$$

by homogeneity,

$$x(\lambda)\delta(t-\lambda) \rightarrow x(\lambda)h(t-\lambda)$$

by additivity, (Note : We can perform additivity on infinite terms only if the sum/integral converges.)

$$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda \longrightarrow \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

This is known as the continuous time convolution of **x(t)** and **h(t)**. This gives the system response **y(t)** to the input **x(t)** in terms of unit impulse response **h(t)**. The convolution of two signals **h(t)** and **x(t)** will be represented symbolically as

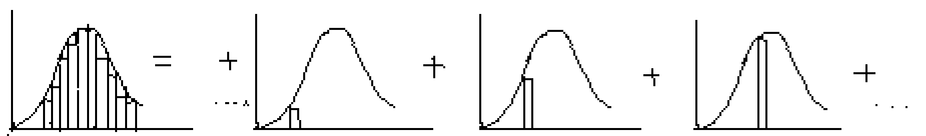
$$y(t) = x(t) * h(t)$$

where as previously seen,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

To explain this graphically,

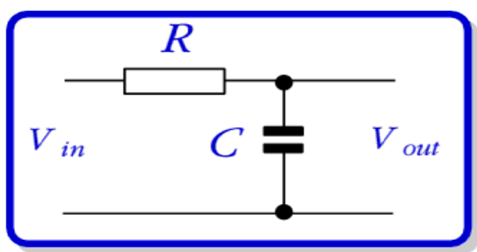
Consider the following input which (as explained above) can be considered to be an approximation of a series of rectangular impulses. And it can be represented using the convolution sum as



Hence, by merely knowing the impulse response one can predict the response of the signal **x(t)** by using the given formula for convolution.

RC System

Consider a RC system consisting of a resistor and a capacitor. We have to find out what the response of this system is to the unit impulse $\delta_\Delta[t]$



First let us understand the response to $\delta_\Delta[t]$

If the input is the unit step function u(t) then the output of the system will be $(1-e^{(-t/RC)})u(t)$. Let us call this output of the system S(t). Hence we can say that the response to $\delta_\Delta[t]$ will be given as follows:

$$\frac{1}{\Delta}\left\{(1-e^{-(t+\frac{\Delta}{2})/\tau})u(t+\frac{\Delta}{2})-(1-e^{-(t-\frac{\Delta}{2})/\tau})u(t-\frac{\Delta}{2})\right\}=R(t)$$

Now as $\Delta \to 0$ the response of $\delta_\Delta[t]$ will be equal to h(t)

Taking limit as $\Delta \to 0$ on both sides and using $\lim_{\Delta \to 0} \frac{1}{\Delta}\left\{S\left(t+\frac{\Delta}{2}\right)-S\left(t-\frac{\Delta}{2}\right)\right\} = \frac{d}{dt}S(t)$ we get

$$h(t) = \frac{d}{dt}S(t)$$

By the sifting property we get $x(t)\delta(t-t_o) = x(t_o)\delta(t-t_o)$

$$h(t) = \frac{1}{\tau}(e^{-t/\tau})u(t)$$

Hence if we are given the ***unit step response*** u(t) we have been able to calculate the continuous ***impulse response*** of the system. Next we shall see how we can get the unit step response from the impulse response of the same system.

Impulse response of RC system

We have seen how we could calculate the impulse response from the unit step response of a system. Now we shall calculate the unit step response, or in general the response to any input signal, given its impulse response. We shall use convolution to obtain the required result.

The unit impulse $\delta(t)$, when fed into the RC system gives the corresponding impulse response h(t), which in this case is given by

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

We shall now find the response to the input signal u(t).

The convolution of an input signal x(t) and the impulse response of a system h(t) is given by the formula:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

But in this case x(t) = u(t), and so the output signal y(t) will be given by :

$$y(t)=\int_{-\infty}^{+\infty} u(\lambda).\frac{1}{\tau}e^{-\left(\frac{t-\lambda}{\tau}\right)}.u(t-\lambda)\ d\lambda$$

Now we have $u(\lambda).u(t-\lambda)=1$ if and only if . $0 \le \lambda \le t$ In all other cases $u(\lambda).u(t-\lambda)=0$

Hence the given equation for y(t) will now simplify to :

$$y(t)=\left\{\int_0^t \frac{1}{\tau}e^{\left(-\frac{t-\lambda}{\tau}\right)}d\lambda\right\}u(t)$$

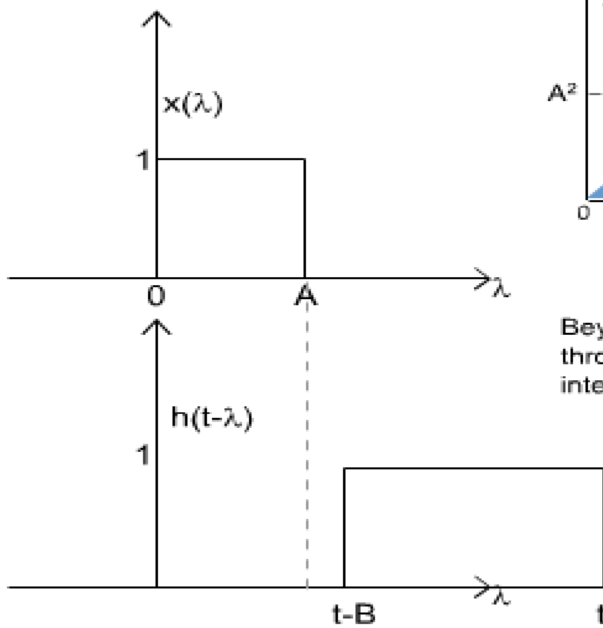
Solving which we get,

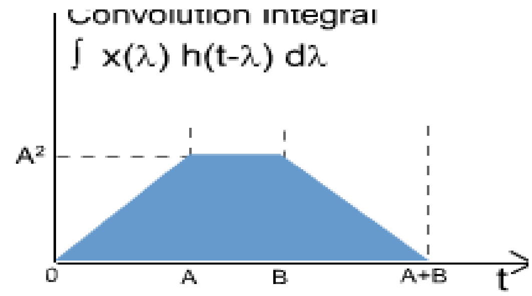
$$y(t)=\left\{1-e^{-t/\tau}\right\}u(t)$$

which is the response to the unit step function.

Hence we have now shown how to calculate the impulse response given the unit step response and also any response given the impulse response. Moreover we can now say that given either the unit step response or the impulse response we can calculate the response to any other input signals.

Convolution Operation

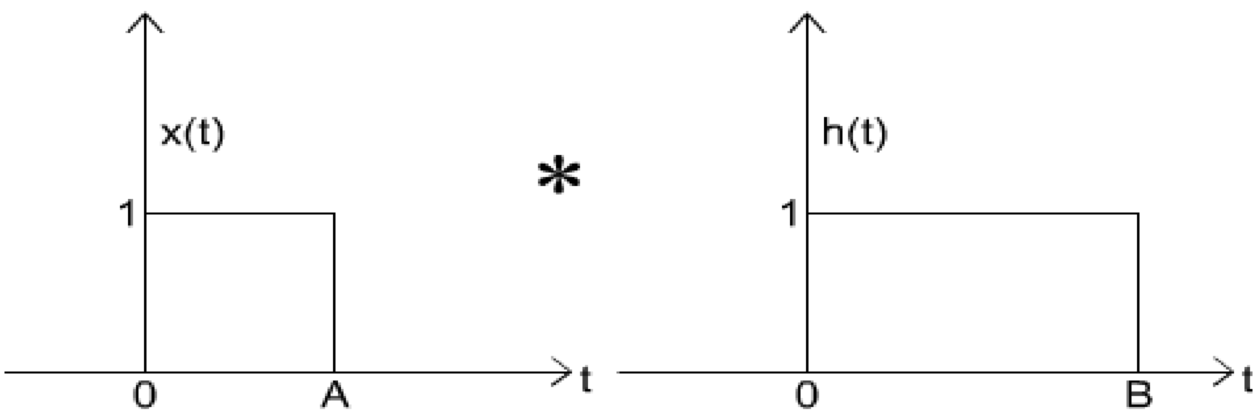




Beyond t=A+B, the overlap again is 0 throughout, and so is the convolution integral.

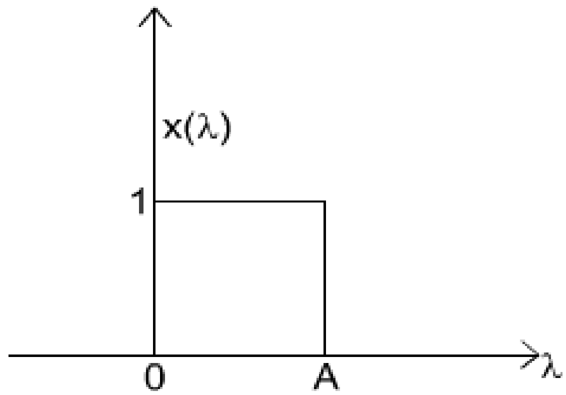
We have thus evaluated the convolution integral over the entire range of 't', and the integral appears as shown in the figure above.

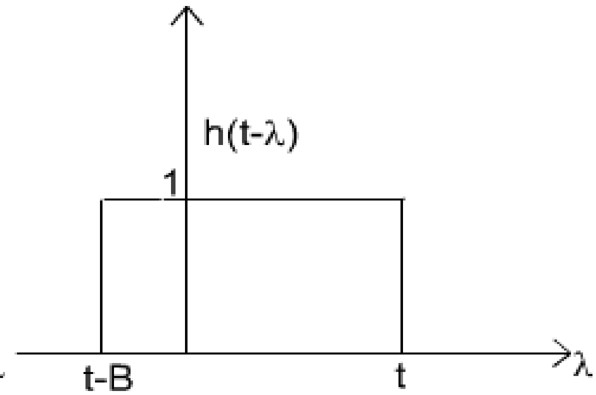
Convolution :



x(t) is to be convolved with h(t)

x(t) and h(t) are both step functions of height 1 and lengths A and B respectively





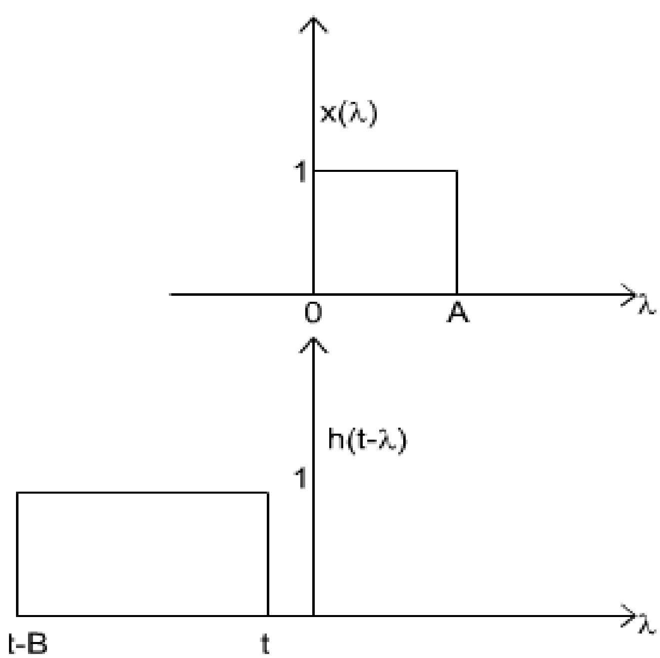
We first find $x(\lambda)$ and $h(t-\lambda)$

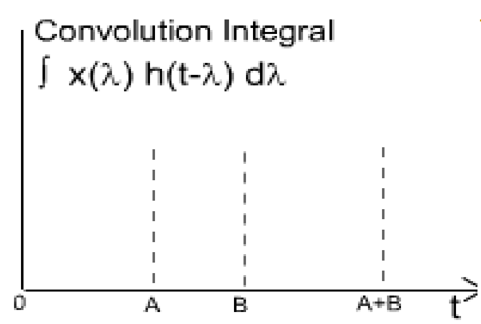
$x(\lambda)$ will be identical to $x(t)$

$h(t-\lambda)$ can be obtained using the procedure explained above

The Convolution integral is given by $\mathbf{y(t) = \int x(\lambda)\, h(t-\lambda)\, d\lambda}$.

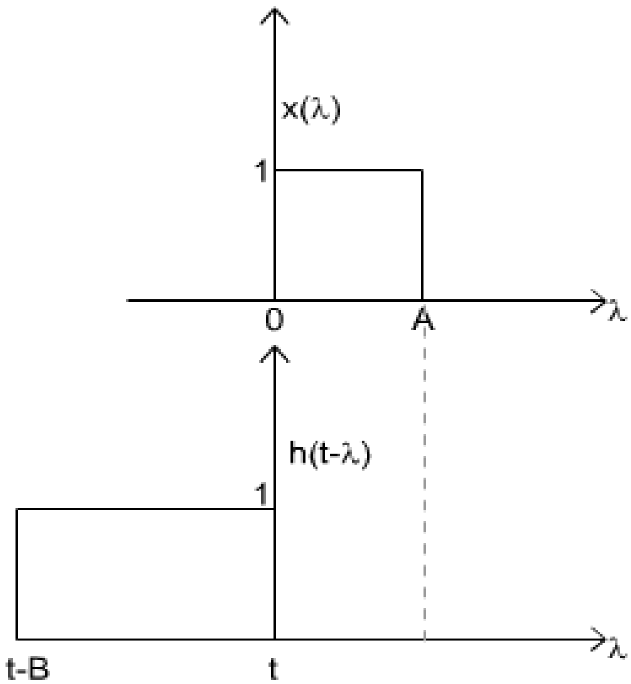
Note that this is an integral in variable λ and 't' is a constant. This integral will give a value for the function $y(t)$ at any instant 't'. Hence, to obtain the entire output signal $y(t)$, we evaluate this integral for every point 't' on which the signal is defined.

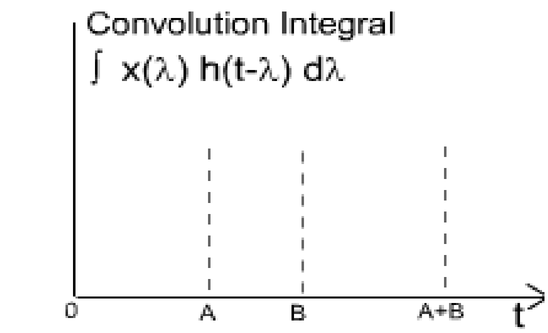




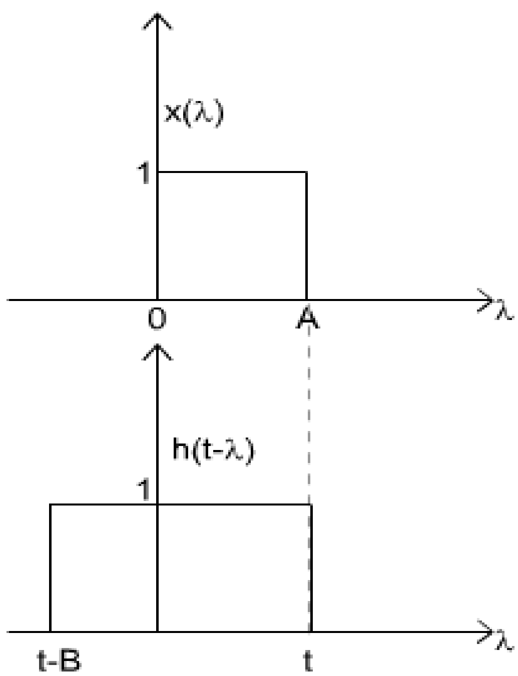
Consider the two functions as two trains. One of the trains, $x(\lambda)$ is stationary, and the other train $h(t-\lambda)$ is a moving train which moves as 't' changes, and λ is a pair of rails on which this train moves.

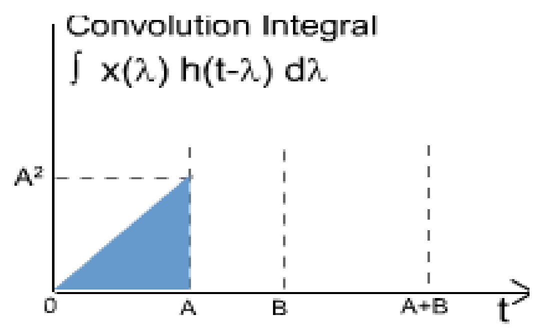
Our convolution integral is going to be the 'exchange' that is taking place between the two trains at every point, or the product of common area between the two functions.



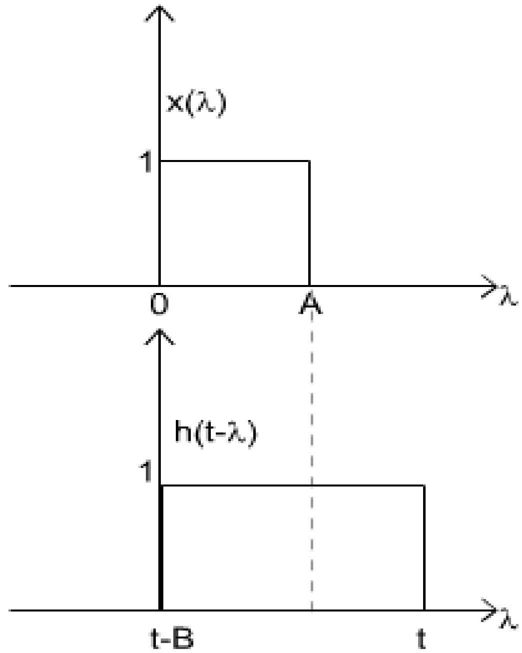


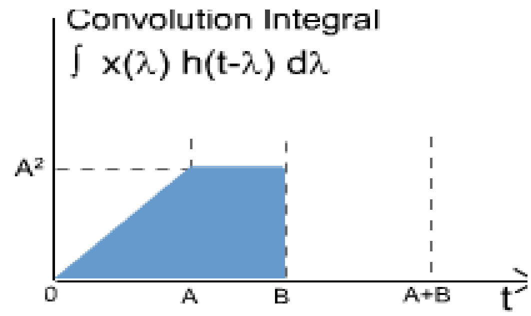
For the region from $t = -\infty$ to $t = 0$, there is no common region between the two functions(or, the function $h(t-\lambda)$ is 0 for all such t. Hence the convolution integral is 0 for all $t < 0$



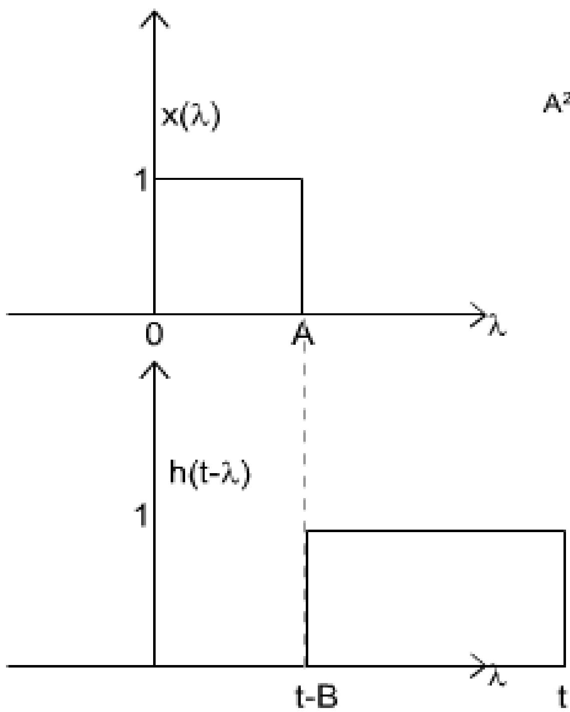


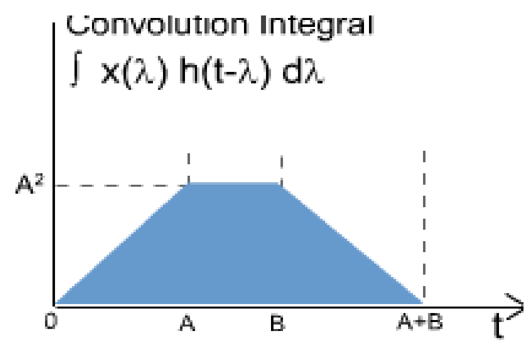
For the region $0 < t < A$, we see that as t increases, the area of overlap also increases accordingly. It can be easily seen that the product of the areas of this overlap follows the function $y(t) = t$





Beyond the point t=A, the overlapping area remains constant till the trailing end of the function h(t-λ) reaches the point t=0, i.e. till the point t=B





As t increases beyond B, we see that the moving function starts 'leaving' the stationary function, or the area of overlap gradually decreases.

This decrease will continue till the point where the two functions do not overlap anymore. As seen in the figure, at t=A+B, this overlap becomes 0

We now interpret the convolution **(x*h)(t)** as the common (shaded) area enclosed under the curves **x(v)** and **h(t-v)** as **v** varies over the entire real axis.

x(v) is the given input function, with the independent variable now called **v**. **h(t-v)** is the impulse response obtained by inverting **h(v)** and then shifting it by **t** units on the **v**-axis.

As **t** increases clearly **h(t-v)** can be considered to be a train moving towards the right, and at each point on the **v**-axis, the area under the product **x(v)** and **h(t-v)** is the value of **y(t)** at that **t**.

Lecture 10: Properties of LTI Systems

Properties of LTI System

In the preceding chapters, we have already derived expressions for discrete as well as continuous time convolution operations.

Discrete :

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

Continuous :

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

We shall now discuss the important properties of convolution for LTI systems.

1) Commutative property:

By the commutative property, the following equations hold true:

a) Discrete time:

$$x[n] * h[n] = h[n] * x[n]$$

Proof: We know that

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

Hence we make the following substitution (n - k = l)

∴ The above expression can be written as

$$y[n] = \sum_{l=-\infty}^{+\infty} x[n-l]\, h[l] \;=\; h[n] * x[n]$$

So it is clear from the derived expression that

$$x[n] * h[n] = h[n] * x[n]$$

Note:

1. 'n' remains constant during the convolution operation so 'n' remains constant in the substitution "n-k = l" even as 'k' and 'l' change.

2. "l" goes from $-\infty$ to $+\infty$, this would not have been so had 'k' been bounded.(e.g :- 0 < k < 11 would make n < l < n – 11)

b) Continuous Time:

$$x(t) * h(t) = h(t) * x(t)$$

Proof:

We know that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Making the substitution $t - \lambda = \phi$, $d\phi = -d\lambda$

Limits:

λ	$-\infty$	∞
ϕ	∞	$-\infty$

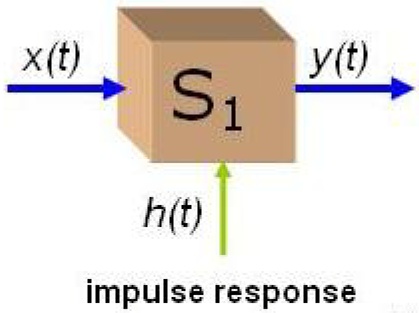
$$\therefore\; y(t) = \int_{-\infty}^{\infty} x(t-\phi)\, h(\phi)\, d\phi$$

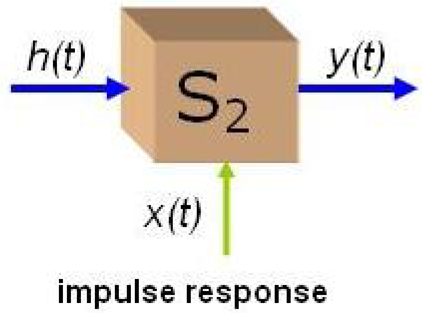
$$= h(t) * x(t)$$

$$\therefore\; x(t) * h(t) = h(t) * x(t)$$

Thus we proved that convolution is commutative in both discrete and continuous variables.

Thus the following two systems: One with input signal x(t) and impulse response h(t) and the other with input signal h(t) and impulse response x(t) both give the same output y(t).





2) Distributive Property:

By this property we mean that convolution is distributive over addition.

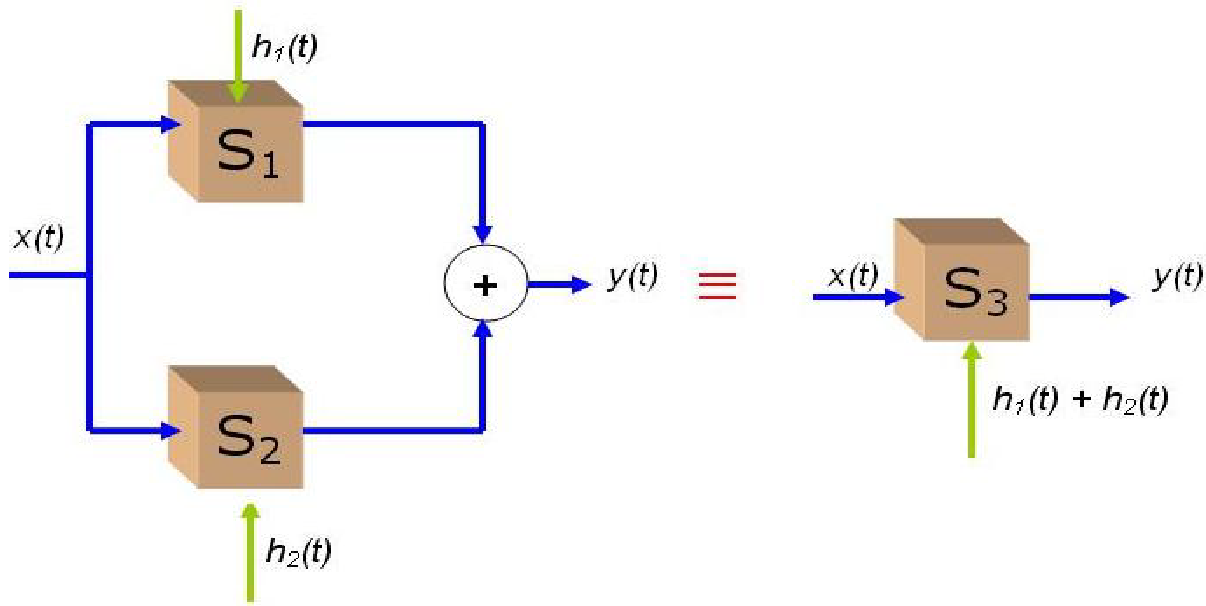
a) Discrete:

$$x[n] * \{ \alpha h_1[n] + \beta h_2[n] \} = \alpha \{ x[n] * h_1[n] \} + \beta \{ x[n] * h_2[n] \} \qquad \alpha, \beta \text{ are constants}$$

b) Continuous:

$$x(t) * \{ \alpha h_1(t) + \beta h_2(t) \} = \alpha \{ x(t) * h_1(t) \} + \beta \{ x(t) * h_2(t) \} \qquad \alpha, \beta \text{ are constants}$$

A parallel combination of LTI systems can be replaced by an equivalent LTI system which is described by the sum of the individual impulse responses in the parallel combination.



3) Associative property

a) Discrete time:

$$y[n] = x[n] * h[n] * g[n]$$

Proof: We know that

$$(x[n] * h_1[n]) * h_2[n] = \sum_{l=-\infty}^{+\infty} (x * h_1)[l] h_2[n-l]$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[k] h_1[l-k] h_2[n-l] \longrightarrow (1)$$

$$x[n] * (h_1[n] * h_2[n]) = \sum_{p=-\infty}^{+\infty} x[p](h_1 * h_2)[n-p]$$

$$= \sum_{p=-\infty}^{+\infty} x[p] \sum_{q=-\infty}^{+\infty} h_1[q] h_2[n-p-q]$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} x[p] h_1[q] h_2[n-p-q] \longrightarrow (2)$$

Making the substitutions: p = k ; q = (l - k) and comparing the two equations makes our proof complete.

Note: As k and l go from $-\infty$ to $+\infty$ independently of each other, so do p and q, however p depends on k, and q depends on l and k.

b) Continuous time :

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

$$\{x(t) * h_1(t)\} * h_2(t) = \int_{-\infty}^{\infty} (x * h_1)(\lambda_1) h_2(t - \lambda_1) d\lambda_1$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda_2) * h_1(\lambda_1 - \lambda_2) h_2(t - \lambda_1) d\lambda_1 d\lambda_2 \rightarrow (1)$$

$$x(t) * \{h_1(t) * h_2(t)\} = \int_{-\infty}^{\infty} x(\lambda_3)(h_1 * h_2)(t - \lambda_3) d\lambda_3$$

$$= \int_{-\infty}^{\infty} x(\lambda_3) \int_{-\infty}^{\infty} h_1(\lambda_4) h_2(t - \lambda_3 - \lambda_4) d\lambda_3 d\lambda_4$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda_3) h_1(\lambda_4) h_2(t - \lambda_3 - \lambda_4) d\lambda_3 d\lambda_4 \rightarrow (2)$$

Lets substitute

$$\lambda_3 = \lambda_2$$

$$\lambda_4 = \lambda_1 - \lambda_2$$

The Jacobian for the above transformation is

$$J = \frac{\partial(\lambda_3, \lambda_4)}{\partial(\lambda_1, \lambda_2)} = 1$$

Doing some further algebra helps us see equation **(2)** transforming into equation **(1)** ,i.e. essentially they are the same. The limits are also the same. Thus the proof is complete.

Implications

This property (Associativity) makes the representation y[n] = x[n]*h[n] *g[n] unambiguous.

From this property, we can conclude that the effective impulse response of a cascaded LTI system is given by the convolution of their individual impulse responses.

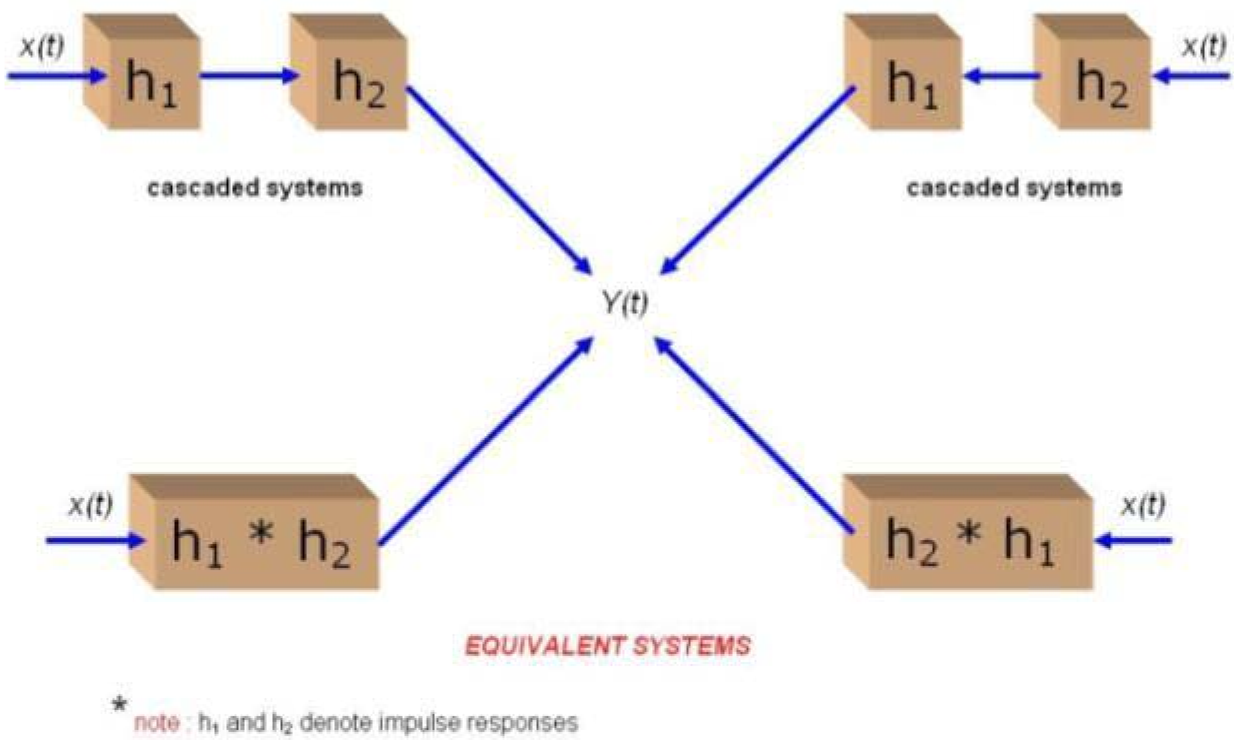


Figure 1.3

Consequently the unit impulse response of a cascaded LTI system is independent of the order in which the individual LTI systems are connected.

Note: All the above three properties are certainly obeyed by LTI systems, but may / may not hold for non-LTI systems in, as seen from the following example:

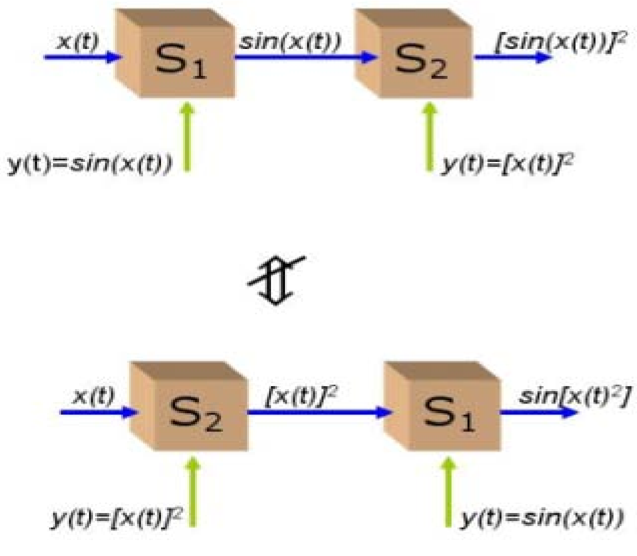


Figure 1.4

4) LTI systems and Memory

Recall that a system is memory less if its output depends on the current input only. From the expression:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

It is easily seen that y[n] depends only on x[n] if and only if $h[n] = 0$ for $n \neq 0$

i.e. $h[n] = k\,\delta[n]$ (k is a constant)

Hence

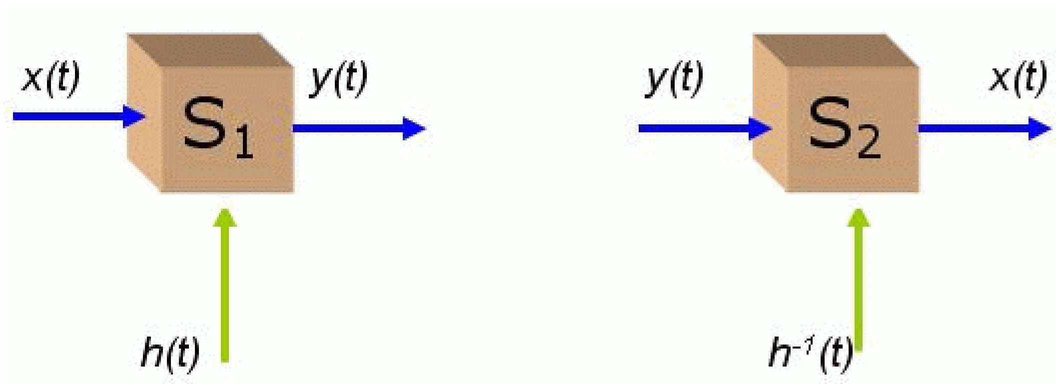
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * k\delta[n] \\ &= k(x * \delta)[n] \\ &= kx[n] \end{aligned}$$

5) Invertibility:

A system is said to be invertible if there exists an inverse system which when connected in series with the original system produces an output identical to the input.

We know that

$$(x * \delta)[n] = x[n]$$
$$(x * h * h^{-1})[n] = x[n]$$
$$(h * h^{-1})[n] = \delta[n]$$



6) Causality:

a) Discrete time:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$ {By Commutative Property}

In order for a discrete LTI system to be causal, y[n] must not depend on x[k] for k > n. For this to be true h[n-k]'s corresponding to the x[k]'s for k > n must be zero. This then requires the impulse response of a causal discrete time LTI system satisfy the following conditions:

$$h[n] = 0 \qquad \forall n < 0$$

Essentially the system output depends only on the past and the present values of the input.

Proof: (By contradiction)

Let in particular h[k] is not equal to 0, for some k<0

$$y[0] = \sum_{k=0}^{\infty} h[k]\, x[-k] \; + \; \sum_{k<0} h[k]\, x[n-k] \; = \; \sum_{k<0} h[k]\, x[n-k]$$ {Refer the eqn. above}

So we need to prove that for all x[n] = 0, n < 0, y[0] = 0

$$h[k] = 2 - j \;\;,\;\; k = -3$$
$$= 1 + j \;\;,\;\; k = -2$$

Now we take a signal defined as

$$x[n-2] = 1 - j \;\;,\;\; n = 2$$
$$x[n-3] = 2 + j \;\;,\;\; n = 3$$

This signal is zero elsewhere. Therefore we get the following result :

$$y[0] = \sum_{k<0} h[k]\, x[n-k]$$
$$= \sum_{k=-2,-3} (1-j)(1+j) \; + \; (2+j)(2-j)$$
$$= 2 + 5 \; = \; 7 \;\; \neq \;\; 0$$

We have come to the result that y[0] ≠0, for the above assumption. ∴ Our assumption stands void. So we conclude that y[n] cannot be independent of x[k] unless h[k] = 0 for k < 0

Note: Here we ensured a non-zero summation by choosing x[n-k]'s as conjugate of h[k]'s.

b) Continuous time:

$$y(t) = \int_{-\infty}^{+\infty} x(v)\, h(t-v)\, dv$$

In order for a continuous LTI system to be causal, y(t) must not depend on x(v) for v > t . For this to be true h(t-v)'s corresponding to the x(v)'s for v > t must be zero.

This then requires the impulse response of a causal continuous time LTI system satisfy the following conditions:

$$h(t) = 0 \qquad \forall\ t < 0$$

As stated before in the discrete time analysis, the system output depends only on the past and the present values of the input.

Proof: (By contradiction)

Suppose, there exists $\alpha > 0$, such that $h(-\alpha) \neq 0$.

Now consider $x(t) = \delta(t - \alpha)$

Since,

$$y(t) = \int_{-\infty}^{+\infty} x(v)\, h(t - v)\, dv$$

$$y(0) = \int_{-\infty}^{+\infty} x(v)\, h(-v)\, dv$$

$$\therefore\ y(0) = h(-\alpha) \neq 0$$

$\Rightarrow$ System is not causal, a contradiction. Hence,

$$h(t) = 0 \quad \forall\ t < 0$$

7) Stability:

A system is said to be stable if its impulse response satisfies the following criterion:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Theorem:

Stability $\Leftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$, in the Discrete domain, OR

Stability $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$, in the Continuous domain.

Proof of sufficiency:

Suppose $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$,

We have $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

If x[n] is bounded i.e. $0 \le |x[n]| \le M_x < \infty \quad \forall\, n$, then:

$$|y[n]| = \left| \sum_k h[k].x[n-k] \right|$$
$$\le \sum_k |h[k]x[n-k]|$$
$$\le \sum_k M_x |h[k]| = M_x \left(\sum_k |h[k]| \right)$$

But as $\sum |h[k]| < \infty \Rightarrow |y[n]| < \infty$

Proof of Necessity:

Take any n.

$$x[n-k] = \frac{\overline{h[k]}}{|h[k]|} \quad \text{if } |h[k]| \ne 0;$$
$$= 0 \quad \text{if } |h[k]| = 0;$$

If | h[k] | = 0, then x[n-k] is bounded with bound 0 $\Rightarrow \sum |h[k]| < \infty$

Then, $y[n] = \sum_k |h[k]| . \frac{\overline{h[k]}}{h[k]|} \quad h[k] \neq 0$

Hence $|y[n]| = \sum |h[k]|$. But since the system is stable $\Rightarrow |y[n]| < \infty$, which in turn implies that $\sum |h[k]| < \infty$

Hence if y[n] is bounded then the condition $\sum |h[k]| < \infty$ must hold.

Hence Proved

A similar proof follows in continuous time when you replace $\sum |h[k]|$ by integral $\int_{-\infty}^{\infty} |h(t)| dt$.