28 Series resonance and Q-factor

At the end of this chapter you should be able to:

- state the conditions for resonance in an a.c. series circuit
- calculate the resonant frequency in an a.c. series circuit,
  \[ f_r = \frac{1}{2\pi \sqrt{LC}} \]
- define Q-factor as \( \frac{X}{R} \) and as \( \frac{V_L}{V} \) or \( \frac{V_C}{V} \)
- determine the maximum value of \( V_C \) and \( V_{COIL} \) and the frequency at which this occurs
- determine the overall Q-factor for two components in series
- define bandwidth and selectivity
- calculate Q-factor and bandwidth in an a.c. series circuit
- determine the current and impedance when the frequency deviates from the resonant frequency

28.1 Introduction

When the voltage \( V \) applied to an electrical network containing resistance, inductance and capacitance is in phase with the resulting current \( I \), the circuit is said to be resonant. The phenomenon of resonance is of great value in all branches of radio, television and communications engineering, since it enables small portions of the communications frequency spectrum to be selected for amplification independently of the remainder.

At resonance, the equivalent network impedance \( Z \) is purely resistive since the supply voltage and current are in phase. The power factor of a resonant network is unity,(i.e., power factor = \( \cos \phi = \cos 0 = 1 \)).

In electrical work there are two types of resonance—one associated with series circuits,(which was introduced in Chapter 15), when the input impedance is a minimum, (which is discussed further in this chapter), and the other associated with simple parallel networks, when the input impedance is a maximum (which is discussed in Chapter 29).

28.2 Series resonance

Figure 28.1 shows a circuit comprising a coil of inductance \( L \) and resistance \( R \) connected in series with a capacitor \( C \). The \( R-L-C \) series circuit has a total impedance \( Z \) given by \( Z = R + j(X_L - X_C) \) ohms, or \( Z = \)
Figure 28.1  R – L – C series circuit

$$V_R (= jR), \quad V_L (= jX_L), \quad V_C (= -jX_C)$$

Figure 28.2  Phasor diagram $|V_L| = |V_C|$

$$R + j(\omega L - 1/\omega C) \text{ ohms where } \omega = 2\pi f. \text{ The circuit is at resonance when } (X_L - X_C) = 0, \text{ i.e., when } X_L = X_C \text{ or } \omega L = 1/(\omega C). \text{ The phasor diagram for this condition is shown in Figure 28.2, where } |V_L| = |V_C|.$$  

Since at resonance $\omega_r L = \frac{1}{\omega_r C}$, $\omega_r^2 = \frac{1}{LC}$ and $\omega = \frac{1}{\sqrt{(LC)}}$

Thus resonant frequency, $f_r = \frac{1}{2\pi \sqrt{(LC)}}$ hertz, since $\omega_r = 2\pi f_r$

Figure 28.3 shows how inductive reactance $X_L$ and capacitive reactance $X_C$ vary with the frequency. At the resonant frequency $f_r$, $|X_L| = |X_C|$. Since impedance $Z = R + j(X_L - X_C)$ and, at resonance, $(X_L - X_C) = 0$, then impedance $Z = R$ at resonance. This is the minimum value possible for the impedance as shown in the graph of the modulus of impedance, $|Z|$, against frequency in Figure 28.4.

At frequencies less than $f_r$, $X_L < X_C$ and the circuit is capacitive; at frequencies greater than $f_r$, $X_L > X_C$ and the circuit is inductive.

Current $I = V/Z$. Since impedance $Z$ is a minimum value at resonance, the current $I$ has a maximum value. At resonance, current $I = V/R$. A graph of current against frequency is shown in Figure 28.4.

Problem 1. A coil having a resistance of 10 Ω and an inductance of 75 mH is connected in series with a 40 µF capacitor across a 200 V a.c. supply. Determine at what frequency resonance occurs, and (b) the current flowing at resonance.
**Problem 2.** An $R$–$L$–$C$ series circuit is comprised of a coil of inductance 10 mH and resistance 8 Ω and a variable capacitor $C$. The supply frequency is 1 kHz. Determine the value of capacitor $C$ for series resonance.

At resonance, $\omega_rL = 1/(\omega_rC)$, from which, capacitance, $C = 1/(\omega_r^2L)$

Hence the capacitance $C = \frac{1}{(2\pi1000)^2(10 \times 10^{-3})} = 2.53 \mu F$

**Problem 3.** A coil having inductance $L$ is connected in series with a variable capacitor $C$. The circuit possesses stray capacitance $C_S$ which is assumed to be constant and effectively in parallel with the variable capacitor $C$. When the capacitor is set to 1000 pF the resonant frequency of the circuit is 92.5 kHz, and when the capacitor is set to 500 pF the resonant frequency is 127.8 kHz. Determine the values of (a) the stray capacitance $C_S$, and (b) the coil inductance $L$.  

**Figure 28.4** $|Z|$ and $I$ plotted against frequency

(a) Resonant frequency,
\[ f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{[(75 \times 10^{-3})(40 \times 10^{-6})]}} \]

i.e., \[ f_r = 91.9 \text{ Hz} \]

(b) Current at resonance, \[ I = \frac{V}{R} = \frac{200}{10} = 20 \text{ A} \]
For a series $R$–$L$–$C$ circuit the resonant frequency $f_r$ is given by:

$$f_r = \frac{1}{2\pi \sqrt{LC}}$$

The total capacitance of $C$ in parallel with $C_S$ is given by $(C + C_S)$

At 92.5 kHz, $C = 1000 \text{ pF}$. Hence

$$92.5 \times 10^3 = \frac{1}{2\pi \sqrt{L(1000 + C_S)10^{-12}}}$$  \hspace{1cm} (1)

At 127.8 kHz, $C = 500 \text{ pF}$. Hence

$$127.8 \times 10^3 = \frac{1}{2\pi \sqrt{L(500 + C_S)10^{-12}}}$$  \hspace{1cm} (2)

(a) Dividing equation (2) by equation (1) gives:

$$\frac{127.8 \times 10^3}{92.5 \times 10^3} = \frac{2\pi \sqrt{L(500 + C_S)10^{-12}}}{2\pi \sqrt{L(1000 + C_S)10^{-12}}}$$

i.e.,

$$\frac{127.8}{92.5} = \frac{\sqrt{L(1000 + C_S)10^{-12}}}{\sqrt{L(500 + C_S)10^{-12}}} = \sqrt{\frac{1000 + C_S}{500 + C_S}}$$

where $C_S$ is in picofarads, from which,

$$\left(\frac{127.8}{92.5}\right)^2 = \frac{1000 + C_S}{500 + C_S}$$

i.e.,

$$1.909 = \frac{1000 + C_S}{500 + C_S}$$

Hence

$$1.909(500 + C_S) = 1000 + C_S$$

$$954.5 + 1.909C_S = 1000 + C_S$$

$$1.909C_S - C_S = 1000 - 954.5$$

$$0.909C_S = 45.5$$

Thus stray capacitance $C_S = 45.5/0.909 = 50 \text{ pF}$

(b) Substituting $C_S = 50 \text{ pF}$ in equation (1) gives:

$$92.5 \times 10^3 = \frac{1}{2\pi \sqrt{L(1050 \times 10^{-12})}}$$
Hence \((92.5 \times 10^3 \times 2\pi)^2 = \frac{1}{L(1050 \times 10^{-12})}\)

from which, \(\text{inductance} \ L = \frac{1}{(1050 \times 10^{-12})(92.5 \times 10^3 \times 2\pi)^2} \ \text{H}\)

\[= 2.82 \ \text{mH}\]

Further problems on series resonance may be found in Section 28.8, problems 1 to 5, page 512

### 28.3 Q-factor

**Q-factor** is a figure of merit for a resonant device such as an \(L–C–R\) circuit.

Such a circuit resonates by cyclic interchange of stored energy, accompanied by energy dissipation due to the resistance.

By definition, at resonance \(Q = 2\pi \left(\frac{\text{maximum energy stored}}{\text{energy loss per cycle}}\right)\)

Since the energy loss per cycle is equal to (the average power dissipated) \(\times\) (periodic time),

\[Q = 2\pi \left(\frac{\text{maximum energy stored}}{\text{average power dissipated} \times \text{periodic time}}\right)\]

\[= 2\pi \left(\frac{\text{maximum energy stored}}{\text{average power dissipated} \times (1/f_r)}\right)\]

since the periodic time \(T = 1/f_r\).

Thus \(Q = 2\pi f_r \left(\frac{\text{maximum energy stored}}{\text{average power dissipated}}\right)\)

i.e., \(Q = \omega_r \left(\frac{\text{maximum energy stored}}{\text{average power dissipated}}\right)\)

where \(\omega_r\) is the angular frequency at resonance.

In an \(L–C–R\) circuit both of the reactive elements store energy during a quarter cycle of the alternating supply input and return it to the circuit source during the following quarter cycle. An inductor stores energy in its magnetic field, then transfers it to the electric field of the capacitor and then back to the magnetic field, and so on. Thus the inductive and capacitive elements transfer energy from one to the other successively with the source of supply ideally providing no additional energy at all. Practical reactors both store and dissipate energy.

Q-factor is an abbreviation for **quality factor** and refers to the ‘goodness’ of a reactive component.
For an inductor, 

\[ Q = \frac{\omega_r \left( \frac{1}{2} L I_m^2 \right)}{I^2 R} = \frac{\omega_r \left( \frac{1}{2} L I_m^2 \right)}{(I_m/\sqrt{2})^2 R} = \frac{\omega_r L}{R} \]  

(28.1)

For a capacitor

\[ Q = \frac{\omega_r \left( \frac{1}{2} C V_m^2 \right)}{(I_m/\sqrt{2})^2 R} = \frac{\omega_r \left( \frac{1}{2} C V_m^2 \right)}{(I_m/\sqrt{2})^2 R} = \frac{\omega_r \frac{1}{2} C I_m^2 (1/\omega_r C)}{(I_m/\sqrt{2})^2 R} \]

i.e.,

\[ Q = \frac{1}{\omega_r C R} \]  

(28.2)

From expressions (28.1) and (28.2) it can be deduced that

\[ Q = \frac{X_L}{R} = \frac{X_C}{R} = \frac{\text{reactance}}{\text{resistance}} \]

In fact, Q-factor can also be defined as

\[ \text{Q-factor} = \frac{\text{reactance power}}{\text{resistance}} = \frac{Q}{P} \]

where \( Q \) is the reactive power which is also the peak rate of energy storage, and \( P \) is the average energy dissipation rate. Hence

\[ \text{Q-factor} = \frac{Q}{P} = \frac{I^2 X_L \text{(or} I^2 X_C)}{I^2 R} = \frac{X_L}{R} \left( \text{or} \frac{X_C}{R} \right) \]

i.e.,

\[ Q = \frac{\text{reactance}}{\text{resistance}} \]

In an \( R-L-C \) series circuit the amount of energy stored at resonance is constant.

When the capacitor voltage is a maximum, the inductor current is zero, and vice versa, i.e., \( \frac{1}{2} L I_m^2 = \frac{1}{2} C V_m^2 \).

Thus the Q-factor at resonance, \( Q_r \), is given by

\[ Q_r = \frac{\omega_r L}{R} = \frac{1}{\omega_r C R} \]  

(28.3)

However, at resonance \( \omega_r = 1/\sqrt{(LC)} \)

Hence \( Q_r = \frac{\omega_r L}{R} = \frac{1}{\sqrt{(LC)}} \left( \frac{L}{R} \right) \)

i.e.,

\[ Q_r = \frac{1}{R} \sqrt{\left( \frac{L}{C} \right)} \]
It should be noted that when Q-factor is referred to, it is nearly always assumed to mean 'the Q-factor at resonance'.

With reference to Figures 28.1 and 28.2, at resonance, \( V_L = V_C \)

\[
V_L = I X_L = I \omega_c L = \frac{V}{R} \omega_c L = \left( \frac{\omega_c L}{R} \right) V = Q_r V
\]

and \( V_C = I X_C = \frac{I}{\omega_c C} = \frac{V}{R} \omega_c C = \left( \frac{1}{\omega_c CR} \right) V = Q_r V \)

Hence, at resonance, \( V_L = V_C = Q_r V \)

or \[
Q_r = \frac{V_L (or \ V_C)}{V}
\]

The voltages \( V_L \) and \( V_C \) at resonance may be much greater than that of the supply voltage \( V \). For this reason \( Q \) is often called the circuit magnification factor. It represents a measure of the number of times \( V_L \) or \( V_C \) is greater than the supply voltage.

The Q-factor at resonance can have a value of several hundreds. Resonance is usually of interest only in circuits of Q-factor greater than about 10; circuits having \( Q \) considerably below this value are effectively merely operating at unity power factor.

Problem 4. A series circuit comprises a 10 \( \Omega \) resistance, a 5 \( \mu \)F capacitor and a variable inductance \( L \). The supply voltage is 20\( \angle 0^\circ \) volts at a frequency of 318.3 Hz. The inductance is adjusted until the p.d. across the 10 \( \Omega \) resistance is a maximum. Determine for this condition (a) the value of inductance \( L \), (b) the p.d. across each component and (c) the Q-factor.

(a) The maximum voltage across the resistance occurs at resonance when the current is a maximum. At resonance, \( \omega_c L = 1/(\omega_c C) \), from which

\[
\text{inductance } L = \frac{1}{\omega_c^2 C} = \frac{1}{(2\pi 318.3)^2(5 \times 10^{-6})} = 0.050 \text{ H or 50 mH}
\]

(b) Current at resonance \( I_r = \frac{V}{R} = \frac{20\angle 0^\circ}{10\angle 0^\circ} = 2.0\angle 0^\circ \) A

p.d. across resistance, \( V_R = I_r R = (2.0\angle 0^\circ)(10) = 20\angle 0^\circ \) V

p.d. across inductance, \( V_L = I X_L \)

\( X_L = 2\pi(318.3)(0.050) = 100 \Omega \)

Hence \( V_L = (2.0\angle 0^\circ)(100\angle 90^\circ) = 200\angle 90^\circ \) V
p.d. across capacitor, \( V_C = IX_C = (2.00∠0°)(100∠-90°) \)
\[ = 200∠-90° \text{ V} \]

(c) Q-factor at resonance, \( Q_r = \frac{V_L}{V} = \frac{200}{20} = 10 \)

\[ \text{Alternatively, } Q_r = \frac{\omega L}{R} = \frac{100}{10} = 10 \]

or
\[ Q_r = \frac{1}{\omega C R} = \frac{1}{2\pi(318.3)(5 \times 10^{-6})(10)} = 10 \]

or
\[ Q_r = \frac{1}{R} \sqrt{\left(\frac{L}{C}\right)} = \frac{1}{10} \sqrt{\left(\frac{0.050}{5 \times 10^{-6}}\right)} = 10 \]

### 28.4 Voltage magnification

For a circuit with a high value of \( Q \) (say, exceeding 100), the maximum volt-drop across the coil, \( V_{COIL} \), and the maximum volt-drop across the capacitor, \( V_C \), coincide with the maximum circuit current at the resonant frequency \( f_r \), as shown in Figure 28.5(a). However, if a circuit of low \( Q \) (say, less than 10) is used, it may be shown experimentally that the maximum value of \( V_C \) occurs at a frequency less than \( f_r \) while the maximum value of \( V_{COIL} \) occurs at a frequency higher than \( f_r \), as shown in Figure 28.5(b). The maximum current, however, still occurs at the resonant frequency with low \( Q \). This is analysed below.

Since \( Q_r = \frac{V_C}{V} \) then \( V_C = V Q_r \)

However \( V_C = IX_C = I \left( \frac{-j}{\omega C} \right) = I \left( \frac{1}{j \omega C} \right) \) and since \( I = \frac{V}{\sqrt{Z}} \),

\[ V_C = \frac{V}{\sqrt{Z}} \left( \frac{1}{j \omega C} \right) = \frac{V}{(j \omega C) \sqrt{Z}} \]

\[ Z = R + j \left( \omega L - \frac{1}{\omega C} \right) \]

thus

\[ V_C = \frac{V}{(j \omega C) \left[ R + j \left( \omega L - \frac{1}{\omega C} \right) \right]} \]

\[ = \frac{V}{j \omega C R + j^2 \omega^2 C L - j \frac{\omega C}{\omega C}} \]

\[ = \frac{V}{(1 - \omega^2 LC) - j \omega CR} \]

\[ = \frac{V[(1 - \omega^2 LC) - j \omega CR]}{[(1 - \omega^2 LC) + j \omega CR][(1 - \omega^2 LC) - j \omega CR]} \]
The magnitude of $V_C$, $|V_C| = \frac{V\sqrt{[1 - \omega^2 LC)^2 + (\omega CR)^2]}}{[1 - \omega^2 LC)^2 + (\omega CR)^2]}$

from the Argand diagram

$$|V_C| = \frac{V}{\sqrt{[1 - \omega^2 LC)^2 + (\omega CR)^2]} \quad (28.4)$$

To find the maximum value of $V_C$, equation (28.4) is differentiated with respect to $\omega$, equated to zero and then solved — this being the normal procedure for maximum/minimum problems. Thus, using the quotient and function of a function rules:

$$\frac{dV_C}{d\omega} = \frac{\sqrt{[1 - \omega^2 LC)^2 + (\omega CR)^2]}[0] - [V]^2(1 - \omega^2 LC)^2}{[(1 - \omega^2 LC)^2 + (\omega CR)^2]} \quad (28.4)$$

$$0 = \frac{V}{2} [2(1 - \omega^2 LC)(-2\omega LC) + 2\omega C^2 R^2]$$

$$= \frac{-V}{2} [2(1 - \omega^2 LC)(-2\omega LC) + 2\omega C^2 R^2]$$

$$= \frac{1}{((1 - \omega^2 LC)^2 + (\omega CR)^2)]^{3/2}} = 0$$

for a maximum value

$$\text{Hence } -\frac{V}{2} [2(1 - \omega^2 LC)(-2\omega LC) + 2\omega C^2 R^2] = 0$$

and $-V[(1 - \omega^2 LC)(-2\omega LC) + \omega C^2 R^2] = 0$

and $$(1 - \omega^2 LC)(-2\omega LC) + \omega C^2 R^2 = 0$$

from which, $\omega C^2 R^2 = (1 - \omega^2 LC)(2\omega LC)$

i.e.,

$$C^2 R^2 = 2LC(1 - \omega^2 LC)$$

$$\frac{C^2 R^2}{2LC} = 2 - 2\omega^2 LC \text{ and } 2\omega^2 LC = 2 - \frac{CR^2}{L}$$

Hence

$$\omega^2 = \frac{2}{2LC} - \frac{CR^2}{2LC} = \frac{1}{LC} - \frac{1}{2} \left( \frac{R}{L} \right)^2$$

$$\text{Figur}$$

28.5 (a) High $Q$-factor

(b) Low $Q$-factor

$$V\text{COL}_{L}$$

$V_C$

$V_H$

$\omega$

$\omega C$

LC

CR

$\omega^2$

$\omega^2 LC$
The resonant frequency, \( \omega_r = \frac{1}{\sqrt{LC}} \) from which, \( \omega_r^2 = \frac{1}{LC} \)

Thus \( \omega^2 = \omega_r^2 \left( 1 - \frac{1}{2Q^2} \right) \) \( \omega \neq \omega_r \) \( \omega \neq \omega_r \)

\[ Q = \frac{\omega_r L}{R} \]

Hence, from equation (28.5) \( \omega^2 = \omega_r^2 \left( 1 - \frac{1}{2Q^2} \right) \)

i.e., \( \omega^2 = \omega_r^2 \left( 1 - \frac{1}{2Q^2} \right) \) \( \omega \neq \omega_r \) \( \omega \neq \omega_r \)

or \( \omega = \omega_r \sqrt{1 - \frac{1}{2Q^2}} \) \( \omega \neq \omega_r \) \( \omega \neq \omega_r \)

or \( f = f_r \sqrt{1 - \frac{1}{2Q^2}} \) \( \omega \neq \omega_r \) \( \omega \neq \omega_r \)

Hence the maximum p.d. across the capacitor does not occur at the resonant frequency, but at a frequency slightly less than \( f_r \) as shown in Figure 28.5(b). If \( Q \) is large, then \( f \approx f_r \) as shown in Figure 28.5(a).

From equation (28.4), \[ |V_C| = \frac{V}{\sqrt{[(1 - \omega^2LC)^2 + (\omega CR)^2]^2}} \]

and substituting \( \omega^2 = \omega_r^2 \left( 1 - \frac{1}{2Q^2} \right) \) from equation (28.6) gives:

maximum value of \( V_C \),

\[ V_{Cm} = \frac{V}{\sqrt{\left( 1 - \omega_r^2 \left( 1 - \frac{1}{2Q^2} \right) LC \right)^2 + \omega_r^2 \left( 1 - \frac{1}{2Q^2} \right) C^2R^2}}} \]

\( \omega_r^2 = \frac{1}{LC} \) hence

\[ V_{Cm} = \frac{V}{\sqrt{\left( 1 - \frac{1}{LC} \left( 1 - \frac{1}{2Q^2} \right) LC \right)^2 + \frac{1}{LC} \left( 1 - \frac{1}{2Q^2} \right) C^2R^2}}} \]
\[ V = \sqrt{\left(1 - \left(1 - \frac{1}{2Q^2}\right)\right)^2 + \frac{CR^2}{L} \left(1 - \frac{1}{2Q^2}\right)} \]

\[ = \frac{V}{\sqrt{\left(\frac{1}{4Q^4} + \frac{CR^2}{L} - \frac{CR^2}{L} \left(1 - \frac{1}{2Q^2}\right)\right)}} \]

(28.8)

\[ Q = \frac{\omega L}{R} = \frac{1}{\omega CR} \text{ hence } Q^2 = \left(\frac{\omega L}{R}\right) \left(\frac{1}{\omega CR}\right) = \frac{L}{CR^2} \]

from which, \( \frac{CR^2}{L} = \frac{1}{Q^2} \)

Substituting in equation (28.8),

\[ V_{cm} = \frac{V}{\sqrt{\left(\frac{1}{4Q^4} + \frac{1}{Q^2} - \frac{1}{2Q^2}\right)}} = \frac{V}{\sqrt{\left(\frac{1}{Q^2} \left[\frac{1}{4Q^4} + 1 - \frac{1}{2Q^2}\right]\right)}} \]

\[ = \frac{V}{Q \sqrt{\left[1 - \frac{1}{4Q^2}\right]}} \]

i.e.,

\[ V_{cm} = \frac{QV}{\sqrt{\left[1 - \left(\frac{1}{2Q}\right)^2\right]}} \]

(28.9)

From equation (28.9), when \( Q \) is large, \( V_{cm} \approx QV \)

If a similar exercise is undertaken for the voltage across the inductor it is found that the maximum value is given by:

\[ V_{lm} = \frac{QV}{\sqrt{\left[1 - \left(\frac{1}{2Q}\right)^2\right]}} \]

i.e., the same equation as for \( V_{cm} \), and frequency,

\[ f = \frac{f_r}{\sqrt{\left[1 - \left(\frac{1}{2Q}\right)^2\right]}} \]
showing that the maximum p.d. across the coil does not occur at the resonant frequency but at a value slightly greater than \( f_r \), as shown in Figure 28.5(b).

**Problem 5.** A series \( L-R-C \) circuit has a sinusoidal input voltage of maximum value 12 V. If inductance, \( L = 20 \text{ mH} \), resistance, \( R = 80 \Omega \), and capacitance, \( C = 400 \text{ nF} \), determine (a) the resonant frequency, (b) the value of the p.d. across the capacitor at the resonant frequency, (c) the frequency at which the p.d. across the capacitor is a maximum, and (d) the value of the maximum voltage across the capacitor.

(a) The resonant frequency,

\[
f_r = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{(20 \times 10^{-3})(400 \times 10^{-9})}} = 1779.4 \text{ Hz}
\]

(b) \( V_C = QV \) and \( Q = \frac{\omega L}{R} \) or \( \frac{1}{\omega CR} \) or \( \frac{1}{R} \sqrt{\frac{L}{C}} \)

Hence \( Q = \frac{(2\pi 1779.4)(20 \times 10^{-3})}{80} = 2.80 \)

Thus \( V_C = QV = (2.80)(12) = 33.60 \text{ V} \)

(c) From equation (28.7), the frequency \( f \) at which \( V_C \) is a maximum value,

\[
f = f_r \sqrt{1 - \frac{1}{2Q^2}} = (1779.4) \sqrt{1 - \frac{1}{2(2.80)^2}} = 1721.7 \text{ Hz}
\]

(d) From equation (28.9), the maximum value of the p.d. across the capacitor is given by:

\[
V_{Cm} = \frac{QV}{\sqrt{1 - \left(\frac{1}{2Q}\right)^2}} = \frac{(2.80)(12)}{\sqrt{1 - \left(\frac{1}{2(2.80)}\right)^2}} = 34.15 \text{ V}
\]

### 28.5 Q-factors in series

If the losses of a capacitor are not considered as negligible, the overall Q-factor of the circuit will depend on the Q-factor of the individual components. Let the Q-factor of the inductor be \( Q_L \) and that of the capacitor be \( Q_C \).
The overall Q-factor, $Q_T = \frac{1}{R_T} \sqrt{\frac{L}{C}}$ from Section (28.3),

where $R_T = R_L + R_C$

Since $Q_L = \frac{\omega c L}{R_L}$ then $R_L = \frac{\omega c L}{Q_L}$ and since

$Q_C = \frac{\omega c C}{R_C}$ then $R_C = \frac{1}{Q_C \omega c}$

Hence $Q_T = \frac{1}{R_L + R_C} \sqrt{\frac{L}{C}} = \sqrt{\frac{L}{C}} \left(\frac{\omega c L}{Q_L} + \frac{1}{Q_C \omega c} \right)$

$$= \frac{1}{\sqrt{Q_L} + \frac{1}{Q_C \sqrt{L/C}}} \sqrt{\frac{L}{C}}$$

since $\omega c = \frac{1}{\sqrt{Q_L}}$

$$= \frac{L}{Q_L \sqrt{L/C} + \frac{L^{1/2} C^{1/2}}{Q_C C}} \sqrt{\frac{L}{C}} = \frac{1}{Q_L \sqrt{L/C} + \frac{L}{Q_C C}} \sqrt{\frac{L}{C}}$$

$$= \frac{1}{Q_L} + \frac{1}{Q_C} = \frac{Q_C + Q_L}{Q_L Q_C}$$

i.e., the overall Q-factor,

$$Q_T = \frac{Q_L Q_C}{Q_L + Q_C}$$

Problem 6. An inductor of Q-factor 60 is connected in series with a capacitor having a Q-factor of 390. Determine the overall Q-factor of the circuit.

From above, overall Q-factor,

$$Q_T = \frac{Q_L Q_C}{Q_L + Q_C} = \frac{(60)(390)}{60 + 390} = \frac{23400}{450} = 52$$

13
28.6 Bandwidth

Figure 28.6 shows how current $I$ varies with frequency $f$ in an $R$–$L$–$C$ series circuit. At the resonant frequency $f_r$, current is a maximum value, shown as $I_r$. Also shown are the points $A$ and $B$ where the current is $0.707$ of the maximum value at frequencies $f_1$ and $f_2$. The power delivered to the circuit is $I^2R$. At $I = 0.707I_r$, the power is $(0.707I_r)^2R = 0.5I_r^2R$, i.e., half the power that occurs at frequency $f_r$. The points corresponding to $f_1$ and $f_2$ are called the half-power points. The distance between these points, i.e., $(f_2 - f_1)$, is called the bandwidth.

When the ratio of two powers $P_1$ and $P_2$ is expressed in decibel units, the number of decibels $X$ is given by:

$$X = 10 \log \left( \frac{P_2}{P_1} \right) \text{ dB}$$

(see Section 10.14, page 126)

Let the power at the half-power points be $0.707I_r^2R$ and let the peak power be $I_r^2R$, then the ratio of the power in decibels is given by:

$$10 \log \left( \frac{I_r^2R}{2} \right) = 10 \log \left( \frac{1}{2} \right) = -3 \text{ dB}$$

It is for this reason that the half-power points are often referred to as ‘the $-3 \text{ dB points}$’.

At the half-power frequencies, $I = 0.707 I_r$, thus impedance

$$Z = \frac{V}{I} = \frac{V}{0.707I_r} = 1.414 \left( \frac{V}{I_r} \right) = \sqrt{2}Z_r = \sqrt{2R}$$

(since at resonance $Z_r = R$)

Since $Z = \sqrt{2R}$, an isosceles triangle is formed by the impedance triangles, as shown in Figure 28.7, where $ab = bc$. From the impedance triangles it can be seen that the equivalent circuit reactance is equal to the circuit resistance at the half-power points.

At $f_1$, the lower half-power frequency $|X_C| > |X_L|$ (see Figure 28.4)

Thus

$$\frac{1}{2\pi f_1L} = 2\pi f_1L = R$$

from which, $1 - 4\pi^2f_1^2LC = 2\pi f_1CR$

e.i., $(4\pi^2LC)f_1^2 + (2\pi CR)f_1 - 1 = 0$

This is a quadratic equation in $f_1$. Using the quadratic formula gives:

$$f_1 = \frac{-(2\pi CR) \pm \sqrt{(2\pi CR)^2 - 4(4\pi^2LC)(-1)}}{2(4\pi^2LC)}$$

$$= \frac{-(2\pi CR) \pm \sqrt{4\pi^2C^2R^2 + 16\pi^2LC}}{8\pi^2LC}$$
\[
\begin{align*}
&= -\frac{(2\pi CR) \pm \sqrt{[4\pi^2C^2(R^2 + (4L/C))]} }{8\pi^2LC} \\
&= -\frac{(2\pi CR) \pm 2\pi C \sqrt{[R^2 + (4L/C)]} }{8\pi^2LC}
\end{align*}
\]

Hence \( f_1 = \frac{-R \pm \sqrt{[R^2 + (4L/C)]} }{4\pi L} = \frac{-R + \sqrt{[R^2 + (4L/C)]} }{4\pi L} \)

(since \( \sqrt{[R^2 + (4L/C)]} > R \) and \( f_1 \) cannot be negative).

At \( f_2 \), the upper half-power frequency \( |X_L| > |X_C| \) (see Figure 28.4)

Thus \( 2\pi f_2L - \frac{1}{2\pi f_2C} = R \)

from which, \( 4\pi^2 f_2^2 LC - 1 = R(2\pi f_2C) \)

i.e., \( (4\pi^2 LC)f_2^2 - (2\pi CR)f_2 - 1 = 0 \)

This is a quadratic equation in \( f_2 \) and may be solved using the quadratic formula as for \( f_1 \), giving:

\( f_2 = \frac{R + \sqrt{[R^2 + (4L/C)]} }{4\pi L} \)

Bandwidth \( = (f_2 - f_1) \)

\[
= \left\{ \frac{R + \sqrt{[R^2 + (4L/C)]} }{4\pi L} \right\} - \left\{ \frac{-R + \sqrt{[R^2 + (4L/C)]} }{4\pi L} \right\} \\
= \frac{2R}{4\pi L} = \frac{R}{2\pi L} = \frac{1}{2\pi L/R} \\
= \frac{f_r}{2\pi f_r/L/R} = \frac{f_r}{Q_r}
\]

from equation (28.3). Hence for a series \( R\!-\!L\!-\!C \) circuit

\[
Q_r = \frac{f_r}{f_2 - f_1} \quad (28.10)
\]

Problem 7. A filter in the form of a series \( L\!-\!R\!-\!C \) circuit is designed to operate at a resonant frequency of 10 kHz. Included within the filter is a 10 mH inductance and 5 \( \Omega \) resistance. Determine the bandwidth of the filter.

Q-factor at resonance is given by

\[
Q_r = \frac{\omega_0L}{R} = \frac{(2\pi \times 10,000)(10 \times 10^{-3})}{5} = 125.66
\]
Since \( Q_r = f_r/(f_2 - f_1) \),

\[
\text{Bandwidth, } (f_2 - f_1) = \frac{f_r}{Q_r} = \frac{10000}{125.66} = 79.6 \text{ Hz}
\]

An alternative equation involving \( f_r \)

At the lower half-power frequency \( f_1 \):

\[
\frac{1}{\omega_1 C} - \omega_1 L = R
\]

At the higher half-power frequency \( f_2 \):

\[
\omega_2 L - \frac{1}{\omega_2 C} = R
\]

Equating gives:

\[
\frac{1}{\omega_1 C} - \omega_1 L = \omega_2 L - \frac{1}{\omega_2 C}
\]

Multiplying throughout by \( C \) gives:

\[
\frac{1}{\omega_1} - \omega_1 LC = \omega_2 LC - \frac{1}{\omega_2}
\]

However, for series resonance, \( \omega_r^2 = 1/(LC) \)

Hence

\[
\frac{1}{\omega_1} - \frac{\omega_1}{\omega_r^2} = \frac{\omega_2}{\omega_r^2} - \frac{1}{\omega_2}
\]

i.e.,

\[
\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{\omega_2}{\omega_1 \omega_2} = \frac{\omega_1 + \omega_2}{\omega_r^2}
\]

Therefore

\[
\frac{\omega_2 + \omega_1}{\omega_1 \omega_2} = \frac{\omega_1 + \omega_2}{\omega_r^2}
\]

from which, \( \omega_r^2 = \omega_1 \omega_2 \) or \( \omega_r = \sqrt{(\omega_1 \omega_2)} \)

Hence

\[
2\pi f_r = \sqrt{[(2\pi f_1)(2\pi f_2)]} \quad \text{and} \quad f_r = \sqrt{(f_1 f_2)}
\]  

(28.11)

**Selectivity** is the ability of a circuit to respond more readily to signals of a particular frequency to which it is tuned than to signals of other frequencies. The response becomes progressively weaker as the frequency departs from the resonant frequency. Discrimination against other signals becomes more pronounced as circuit losses are reduced, i.e., as the Q-factor is increased. Thus \( Q_r = f_r/(f_2 - f_1) \) is a measure of the circuit selectivity in terms of the points on each side of resonance where the circuit current has fallen to 0.707 of its maximum value reached at resonance. The higher the Q-factor, the narrower the bandwidth and the more selective is the circuit. Circuits having high Q-factors (say, in the order 300) are therefore useful in communications engineering. A high Q-factor in a series power circuit has disadvantages in that it can lead to dangerously high voltages across the insulation and may result in electrical breakdown.

For example, suppose that the working voltage of a capacitor is stated as 1 kV and is used in a circuit having a supply voltage of 240 V. The maximum value of the supply will be \( \sqrt{2}(240) \), i.e., 340 V. The working voltage of the capacitor would appear to be ample. However, if the Q-factor is, say, 10, the voltage across the capacitor will reach 2.4 kV.
Since the capacitor is rated only at 1 kV, dielectric breakdown is more than likely to occur.

Low Q-factors, say, in the order of 5 to 25, may be found in power transformers using laminated iron cores.

A capacitor-start induction motor, as used in domestic appliances such as washing machines and vacuum-cleaners, having a Q-factor as low as 1.5 at starting would result in a voltage across the capacitor 1.5 times that of the supply voltage; hence the cable joining the capacitor to the motor would require extra insulation.

Problem 8. An $R-L-C$ series circuit has a resonant frequency of 1.2 kHz and a Q-factor at resonance of 30. If the impedance of the circuit at resonance is 50 Ω determine the values of (a) the inductance, and (b) the capacitance. Find also (c) the bandwidth, (d) the lower and upper half-power frequencies and (e) the value of the circuit impedance at the half-power frequencies.

(a) At resonance the circuit impedance, $Z = R$, i.e., $R = 50 \Omega$.

Q-factor at resonance, $Q_r = \omega_r L / R$

Hence inductance, $L = \frac{Q_r R}{\omega_r} = \frac{30(50)}{(2\pi 1200)} = 0.199 \text{ H or } 199 \text{ mH}$

(b) At resonance $\omega_r L = 1 / (\omega_r C)$

Hence capacitance, $C = \frac{1}{\omega_r^2 L} = \frac{1}{(2\pi 1200)^2(0.199)}$

$= 0.088 \mu\text{F or } 88 \text{ nF}$

(c) Q-factor at resonance is also given by $Q_r = f_r / (f_2 - f_1)$, from which,

bandwidth, $(f_2 - f_1) = \frac{f_r}{Q_r} = \frac{1200}{30} = 40 \text{ Hz}$

(d) From equation (28.11), resonant frequency, $f_r = \sqrt{(f_1 f_2)}$, i.e., $1200 = \sqrt{(f_1 f_2)}$

from which, $f_1 f_2 = (1200)^2 = 1.44 \times 10^6$ (28.12)

From part(c), $f_2 - f_1 = 40$ (28.13)

From equation (28.12), $f_1 = (1.44 \times 10^6) / f_2$

Substituting in equation (28.13) gives:

$f_2 - \frac{1.44 \times 10^6}{f_2} = 40$
Multiplying throughout by \( f_2 \) gives:

\[
f_2^2 - 1.44 \times 10^6 = 40 f_2
\]
i.e., \( f_2^2 - 40 f_2 - 1.44 \times 10^6 = 0 \)

This is a quadratic equation in \( f_2 \). Using the quadratic formula gives:

\[
f_2 = \frac{40 \pm \sqrt{(40)^2 - 4(1.44 \times 10^6)}}{2} = \frac{40 \pm 2400}{2}
\]

\[
= \frac{40 + 2400}{2} \text{ (since } f_2 \text{ cannot be negative)}
\]

Hence the upper half-power frequency, \( f_2 = 1220 \text{ Hz} \)

From equation (28.12), the lower half-power frequency,

\[
f_1 = f_2 - 40 = 1220 - 40 = 1180 \text{ Hz}
\]

Note that the upper and lower half-power frequency values are symmetrically placed about the resonance frequency. This is usually the case when the Q-factor has a high value (say, >10).

(e) At the half-power frequencies, current \( I = 0.707 I_r \)

Hence impedance,

\[
Z = \frac{V}{I} = \frac{V}{0.707 I_r} = 1.414 \left( \frac{V}{I_r} \right) = \sqrt{2} Z_r = \sqrt{2} R
\]

Thus impedance at the half-power frequencies,

\[
Z = \sqrt{2} R = \sqrt{2} (50) = 70.71 \Omega
\]

Problem 9. A series \( R-L-C \) circuit is connected to a 0.2 V supply and the current is at its maximum value of 4 mA when the supply frequency is adjusted to 3 kHz. The Q-factor of the circuit under these conditions is 100. Determine the value of (a) the circuit resistance, (b) the circuit inductance, (c) the circuit capacitance, and (d) the voltage across the capacitor

Since the current is at its maximum, the circuit is at resonance and the resonant frequency is 3 kHz.

(a) At resonance, impedance, \( Z = R = \frac{V}{I} = \frac{0.2}{4 \times 10^{-3}} = 50 \Omega \)

Hence the circuit resistance in 50 \( \Omega \)

(b) Q-factor at resonance is given by \( Q_r = \frac{\omega_r L}{R} \), from which,

inductance, \( L = \frac{Q_r R}{\omega_r} = \frac{(100)(50)}{2\pi 3000} = 0.265 \text{ H or 265 mH} \)
(c) Q-factor at resonance is also given by \( Q_r = 1/(\omega_r CR) \), from which,
\[
C = \frac{1}{\omega_r R Q_r} = \frac{1}{2\pi 3000(50)(100)} = 0.0106 \, \mu F \text{ or } 10.6 \, nF
\]

(d) Q-factor at resonance in a series circuit represents the voltage magnification, i.e., \( Q_r = V_C/V \), from which, \( V_C = Q_r V \approx (100)(0.2) = 20 \, V \).

Hence the voltage across the capacitor is 20 V

(Alternatively, \( V_C = IX_C = I \omega r C = \frac{4 \times 10^{-3}}{2\pi 3000(0.0106 \times 10^{-6})} = 20 \, V \))

Problem 10. A coil of inductance 351.8 mH and resistance 8.84 Ω is connected in series with a 20 µF capacitor. Determine (a) the resonant frequency, (b) the Q-factor at resonance, (c) the bandwidth, and (d) the lower and upper −3dB frequencies.

(a) Resonant frequency, \( f_r = \frac{1}{2\pi \sqrt{(LC)}} = \frac{1}{2\pi \sqrt{(0.3518)(20 \times 10^{-6})}} \]
\( = 60.0 \, Hz \)

(b) Q-factor at resonance, \( Q_r = \frac{1}{\omega_r} \sqrt{\frac{L}{C}} = \frac{1}{8.84} \sqrt{\frac{0.3518}{20 \times 10^{-6}}} = 15 \)

[Alternatively, \( Q_r = \frac{\omega_r L}{R} = \frac{2\pi(60.0)(0.3518)}{8.84} = 15 \)

or \( Q_r = \frac{1}{\omega_r CR} = \frac{1}{2\pi 60.0(20 \times 10^{-6})(8.84)} = 15 \)]

(c) Bandwidth, \( (f_2 - f_1) = \frac{f_s}{Q_r} = \frac{60.0}{15} = 4 \, Hz \)

(d) With a Q-factor of 15 it may be assumed that the lower and upper −3 dB frequencies, \( f_1 \) and \( f_2 \) are symmetrically placed about the resonant frequency of 60.0 Hz. Hence the lower −3 dB frequency, \( f_1 = 58 \, Hz \), and the upper −3 dB frequency, \( f_2 = 62 \, Hz \).

[This may be checked by using \( (f_2 - f_1) = 4 \) and \( f_r = \sqrt{(f_1 f_2)} \)]

28.7 Small deviations from the resonant frequency

Let \( \omega_1 \) be a frequency below the resonant frequency \( \omega_r \) in an \( L–R–C \) series circuit, and \( \omega_2 \) be a frequency above \( \omega_r \) by the same amount as \( \omega_1 \) is below, i.e., \( \omega_r - \omega_1 = \omega_2 - \omega_r \)
Let the fractional deviation from the resonant frequency be $\delta$ where
\[
\delta = \frac{\omega_r - \omega_1}{\omega_r} = \frac{\omega_2 - \omega_r}{\omega_r}
\]
Hence $\omega_1 = \omega_r(1 - \delta)$ and $\omega_2 = \omega_r + \omega_r \delta$
from which, $\omega_1 = \omega_r - \omega_r \delta$ and $\omega_2 = \omega_r + \omega_r \delta$
i.e., $\omega_1 = \omega_r(1 - \delta)$ \hspace{1cm} (28.14)
and $\omega_2 = \omega_r(1 + \delta)$ \hspace{1cm} (28.15)
At resonance, $I_r = \frac{V}{R}$, and at other frequencies, $I = \frac{V}{Z}$ where $Z$ is the circuit impedance.
Hence
\[
\frac{I}{I_r} = \frac{V/Z}{V/R} = \frac{R}{Z} = \frac{R}{R + j \left(\omega_L - \frac{1}{\omega_C}\right)}
\]
From equation (28.15), at frequency $\omega_2$, 
\[
\frac{I}{I_r} = R
\]
\[
= R + j \left[\frac{\omega_L(1 + \delta)}{R(1 + \delta)} - \frac{1}{\omega_RC(1 + \delta)}\right]
\]
At resonance, $\frac{1}{\omega_R C} = \omega_r L$ hence
\[
\frac{I}{I_r} = \frac{1}{1 + j \left[\frac{\omega_L}{R(1 + \delta)} - \frac{\omega_L}{R(1 + \delta)}\right]}
\]
\[
= \frac{1}{1 + j \left[\frac{\omega_L}{R}(1 + \delta) - \frac{1}{(1 + \delta)}\right]}
\]
Since $\frac{\omega_L}{R} = Q$ then
\[
\frac{I}{I_r} = \frac{1}{1 + jQ \left[\frac{(1 + \delta)^2 - 1}{(1 + \delta)}\right]} = \frac{1}{1 + jQ \left[\frac{1 + 2\delta + \delta^2 - 1}{(1 + \delta)}\right]}
\]
\[
= \frac{1}{1 + jQ \left[\frac{2\delta + \delta^2}{1 + \delta}\right]} = \frac{1}{1 + j\delta Q \left[\frac{2 + \delta}{1 + \delta}\right]}
\]
If the deviation from the resonant frequency $\delta$ is very small such that $\delta \ll 1$

then

$$\frac{I}{I_r} \approx \frac{1}{1 + j\delta \Omega} = \frac{1}{1 + j2\delta \Omega}$$ (28.16)

and

$$\frac{I}{I_r} = \frac{V/Z}{V/Z_r} = \frac{Z_r}{Z} = \frac{1}{1 + j2\delta \Omega}$$

from which,

$$\frac{Z}{Z_r} = 1 + j2\delta \Omega$$ (28.17)

It may be shown that at frequency $\omega_1$, $\frac{I}{I_r} = \frac{1}{1 - j2\delta \Omega}$ and

$$\frac{Z}{Z_r} = 1 - j2\delta \Omega$$

Problem 11. In an $L$–$R$–$C$ series network, the inductance, $L = 8$ mH, the capacitance, $C = 0.3$ $\mu$F, and the resistance, $R = 15$ $\Omega$. Determine the current flowing in the circuit when the input voltage is 7.5$\angle0^\circ$ V and the frequency is (a) the resonant frequency, (b) a frequency 3% above the resonant frequency. Find also (c) the impedance of the circuit when the frequency is 3% above the resonant frequency.

(a) At resonance, $Z_r = R = 15$ $\Omega$

Current at resonance, $I_r = \frac{V}{Z_r} = \frac{7.5\angle0^\circ}{15\angle0^\circ} = 0.5\angle0^\circ$ A

(b) If the frequency is 3% above the resonant frequency, then $\delta = 0.03$

From equation (28.16), $\frac{I}{I_r} = \frac{1}{1 + j2\delta \Omega}$

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{1}{15} \sqrt{\frac{8 \times 10^{-3}}{0.3 \times 10^{-6}}} = 10.89$$

Hence

$$\frac{I}{0.5\angle0^\circ} = \frac{1}{1 + j2(0.03)(10.89)} = \frac{1}{1 + j0.6534}$$

$$= \frac{1}{1.1945\angle33.16^\circ}$$

and

$$I = \frac{0.5\angle0^\circ}{1.1945\angle33.16^\circ} = 0.4186\angle-33.16^\circ$$ A

(c) From equation (28.17), $\frac{Z}{Z_r} = 1 + j2\delta \Omega$
hence  \[ Z = Z(1 + j2\delta Q) = R(1 + j2\delta Q) \]
\[ = 15(1 + j2(0.03)(10.89)) \]
\[ = 15(1 + j0.6534) \]
\[ = 15(1.1945\angle33.16^\circ) \]
\[ = 17.92\angle33.16^\circ \ \Omega \]
Alternatively, \[ Z = \frac{V}{I} = \frac{7.5\angle0^\circ}{0.4186\angle-33.16^\circ} = 17.92\angle33.16^\circ \ \Omega \]

---

**28.8 Further problems on series resonance and Q-factor**

### Series resonance

1. A coil having an inductance of 50 mH and resistance 8.0 \( \Omega \) is connected in series with a 25 \( \mu \)F capacitor across a 100 V a.c. supply. Determine (a) the resonant frequency of the circuit, and (b) the current flowing at resonance.  
   
   [(a) 142.4 Hz (b) 12.5 A]

2. The current at resonance in a series \( R-L-C \) circuit is 0.12 mA. The circuit has an inductance of 0.05 H and the supply voltage is 24 mV at a frequency of 40 kHz. Determine (a) the circuit resistance, and (b) the circuit capacitance.  
   
   [(a) 200 \( \Omega \) (b) 316.6 pF]

3. A coil of inductance 2.0 mH and resistance 4.0 \( \Omega \) is connected in series with a 0.3 \( \mu \)F capacitor. The circuit is connected to a 5.0 V, variable frequency supply. Calculate (a) the frequency at which resonance occurs, (b) the voltage across the capacitance at resonance, and (c) the voltage across the coil at resonance.  
   
   [(a) 6.50 kHz (b) 102.1 V (c) 102.2 V]

4. A series \( R-L-C \) circuit having an inductance of 0.40 H has an instantaneous voltage, \( v = 60\sin(4000t - (\pi/6)) \) volts and an instantaneous current, \( i = 2.0\sin 4000t \) amperes. Determine (a) the values of the circuit resistance and capacitance, and (b) the frequency at which the circuit will be resonant.  
   
   [(a) 26 \( \Omega \); 154.8 nF (b) 639.6 Hz]

5. A variable capacitor \( C \) is connected in series with a coil having inductance \( L \). The circuit possesses stray capacitance \( C_s \) which is assumed to be constant and effectively in parallel with the variable capacitor \( C \). When the capacitor is set to 2.0 nF the resonant frequency of the circuit is 86.85 kHz, and when the capacitor is set to 1.0 nF the resonant frequency is 120 kHz. Determine the values of (a) the stray circuit capacitance \( C_s \), and (b) the coil inductance \( L \).  
   
   [(a) 100 pF (b) 1.60 mH]
Q-factor and bandwidth

6 A series $R-L-C$ circuit comprises a $5 \mu F$ capacitor, a $4 \Omega$ resistor and a variable inductance $L$. The supply voltage is $1000 V$ at a frequency of $159.1 Hz$. The inductance is adjusted until the p.d. across the $4 \Omega$ resistance is a maximum. Determine for this condition (a) the value of inductance, (b) the p.d. across each component, and (c) the Q-factor of the circuit.

\[(a) 200 \text{ mH} \quad (b) \quad V_R = 1000V \quad V_L = 500\angle 90^\circ V \quad V_C = 500\angle -90^\circ V \quad (c) 50\]

7 A coil of resistance $10.05 \Omega$ and inductance $400 \text{ mH}$ is connected in series with a $0.396 \mu F$ capacitor. Determine (a) the resonant frequency, (b) the resonant Q-factor, (c) the bandwidth, and (d) the lower and upper half-power frequencies.

\[(a) 400 \text{ Hz} \quad (b) 100 \text{ Hz} \quad (c) 4 \text{ Hz} \quad (d) 398 \text{ Hz and 402 Hz}\]

8 An $R-L-C$ series circuit has a resonant frequency of $2 \text{ kHz}$ and a Q-factor at resonance of $40$. If the impedance of the circuit at resonance is $30 \Omega$ determine the values of (a) the inductance and (b) the capacitance. Find also (c) the bandwidth, (d) the lower and upper $\pm 3 \text{ dB}$ frequencies, and (e) the impedance at the $\pm 3 \text{ dB}$ frequencies.

\[(a) 95.5 \text{ mH} \quad (b) 66.3 \text{ nF} \quad (c) 50 \text{ Hz} \quad (d) 1975 \text{ Hz and 2025 Hz} \quad (e) 42.43 \Omega\]

9 A filter in the form of a series $L-C-R$ circuit is designed to operate at a resonant frequency of $20 \text{ kHz}$ and incorporates a $20 \text{ mH}$ inductor and $30 \Omega$ resistance. Determine the bandwidth of the filter.

\[238.7 \text{ Hz}\]

10 A series $L-R-C$ circuit has a supply input of $5 \text{ volts}$. Given that inductance, $L = 5 \text{ mH}$, resistance, $R = 75 \Omega$ and capacitance, $C = 0.2 \mu F$, determine (a) the resonant frequency, (b) the value of voltage across the capacitor at the resonant frequency, (c) the frequency at which the p.d. across the capacitance is a maximum, and (d) the value of the maximum voltage across the capacitor.

\[(a) 5033 \text{ Hz} \quad (b) 10.54 \text{ V} \quad (c) 4741 \text{ Hz} \quad (d) 10.85 \text{ V}\]

11 A capacitor having a Q-factor of $250$ is connected in series with a coil which has a Q-factor of $80$. Calculate the overall Q-factor of the circuit.

\[60.61\]

12 An $R-L-C$ series circuit has a maximum current of $2 \text{ mA}$ flowing in it when the frequency of the $0.1 \text{ V}$ supply is $4 \text{ kHz}$. The Q-factor of the circuit under these conditions is $90$. Determine (a) the voltage across the capacitor, and (b) the values of the circuit resistance, inductance and capacitance.

\[(a) 9 \text{ V} \quad (b) 50 \Omega \quad 0.179 \text{ H} \quad 8.84 \text{ nF}\]

13 Calculate the inductance of a coil which must be connected in series with a $4000 \text{ pF}$ capacitor to give a resonant frequency of $200 \text{ kHz}$. If the coil has a resistance of $12 \Omega$, determine the circuit Q-factor.

\[158.3 \mu H; \quad 16.58\]
14 A circuit consists of a coil of inductance 200 $\mu$H and resistance 8.0 $\Omega$ in series with a lossless 500 pF capacitor. Determine (a) the resonant Q-factor, and (b) the bandwidth of the circuit.

[(a) 79.06 (b) 6366 Hz]

15 A coil of inductance 200 $\mu$H and resistance 50.27 $\Omega$ and a variable capacitor are connected in series to a 5 mV supply of frequency 2 MHz. Determine (a) the value of capacitance to tune the circuit to resonance, (b) the supply current at resonance, (c) the p.d. across the capacitor at resonance, (d) the bandwidth, and (e) the half-power frequencies.

[(a) 31.66 pF (b) 99.46 $\mu$A (c) 250 mV (d) 40 kHz (e) 2.02 MHz; 1.98 MHz]

16 A supply voltage of 3 V is applied to a series $R$–$L$–$C$ circuit whose resistance is 12 $\Omega$, inductance is 7.5 mH and capacitance is 0.5 $\mu$F. Determine (a) the current flowing at resonance, (b) the current flowing at a frequency 2.5% below the resonant frequency, and (c) the impedance of the circuit when the frequency is 1% lower than the resonant frequency.

[(a) 0.25 A (b) 0.223$\angle$27.04$^\circ$A (c) 13.47$\angle$27.04$^\circ$ $\Omega$]
29 Parallel resonance and Q-factor

At the end of this chapter you should be able to:

- state the condition for resonance in an a.c. parallel network
- calculate the resonant frequency in a.c. parallel networks
- calculate dynamic resistance \( R_D = \frac{L}{CR} \) in an a.c. parallel network
- calculate Q-factor and bandwidth in an a.c. parallel network
- determine the overall Q-factor for capacitors connected in parallel
- determine the impedance when the frequency deviates from the resonant frequency

29.1 Introduction

A parallel network containing resistance \( R \), pure inductance \( L \) and pure capacitance \( C \) connected in parallel is shown in Figure 29.1. Since the inductance and capacitance are considered as pure components, this circuit is something of an ‘ideal’ circuit. However, it may be used to highlight some important points regarding resonance which are applicable to any parallel circuit.

From Figure 29.1,

- the admittance of the resistive branch, \( G = \frac{1}{R} \)
- the admittance of the inductive branch, \( B_L = \frac{1}{jX_L} = -\frac{j}{\omega L} \)
- the admittance of the capacitive branch, \( B_C = \frac{1}{-jX_C} = \frac{j}{1/\omega C} = j\omega C \)

Total circuit admittance, \( Y = G + j(B_C - B_L) \),

i.e.,

\[
Y = \frac{1}{R} + j \left( \omega C - \frac{1}{\omega L} \right)
\]

The circuit is at resonance when the imaginary part is zero, i.e., when \( \omega C - (1/\omega L) = 0 \). Hence at resonance \( \omega_C = 1/(\omega L) \) and \( \omega_C^2 = 1/(LC) \).
Figure 29.2  $|Y|$ plotted against frequency

from which $\omega_r = 1/\sqrt{(LC)}$ and the resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{(LC)}} \text{ hertz}$$

the same expression as for a series $R–L–C$ circuit.

Figure 29.2 shows typical graphs of $B_C$, $B_L$, $G$ and $Y$ against frequency $f$ for the circuit shown in Figure 29.1. At resonance, $B_C = B_L$ and admittance $Y = G = 1/R$. This represents the condition of minimum admittance for the circuit and thus maximum impedance.

Since current $I = V/Z = VY$, the current is at a minimum value at resonance in a parallel network.

From the ideal circuit of Figure 29.1 we have therefore established the following facts which apply to any parallel circuit. At resonance:

(i) admittance $Y$ is a minimum
(ii) impedance $Z$ is a maximum
(iii) current $I$ is a minimum
(iv) an expression for the resonant frequency $f_r$ may be obtained by making the ‘imaginary’ part of the complex expression for admittance equal to zero.

29.2 The $LR–C$ parallel network

A more practical network, containing a coil of inductance $L$ and resistance $R$ in parallel with a pure capacitance $C$, is shown in Figure 29.3.

Admittance of coil, $Y_{COIL} = \frac{1}{R + jX_L}$

$$= \frac{R - jX_L}{R^2 + X_L^2}$$

$$= \frac{R}{R^2 + \omega^2L^2} - \frac{j\omega L}{R^2 + \omega^2L^2}$$

Admittance of capacitor, $Y_C = \frac{1}{-jX_C}$

$$= \frac{j}{X_C} = j\omega C$$

Total circuit admittance, $Y = Y_{COIL} + Y_C$

$$= \frac{R}{R^2 + \omega^2L^2} - \frac{j\omega L}{R^2 + \omega^2L^2} + j\omega C \quad (29.1)$$

At resonance, the total circuit admittance $Y$ is real ($Y = R/(R^2 + \omega^2L^2)$), i.e., the imaginary part is zero. Hence, at resonance:

$$-\frac{\omega L}{R^2 + \omega^2L^2} + \omega L C = 0$$
Therefore \( \omega_r L = \omega_0 C \) and \( \frac{L}{C} = R^2 + \omega_0^2 L^2 \)

Thus \( \omega_r^2 L^2 = \frac{L}{C} - R^2 \)

and \( \omega_r^2 = \frac{L}{CL^2} - \frac{R^2}{L^2} = \frac{1}{LC} - \frac{R^2}{L^2} \) \( \tag{29.2} \)

Hence \( \omega_r = \sqrt{\left( \frac{1}{LC} - \frac{R^2}{L^2} \right)} \)

and resonant frequency, \( f_r = \frac{1}{2\pi} \sqrt{\left( \frac{1}{LC} - \frac{R^2}{L^2} \right)} \) \( \tag{29.3} \)

Note that when \( R^2/L^2 \ll 1/(LC) \) then \( f_r = 1/(2\pi \sqrt{LC}) \), as for the series \( R-L-C \) circuit. Equation (29.3) is the same as obtained in Chapter 16, page 248; however, the above method may be applied to any parallel network as demonstrated in Section 29.4 below.

### 29.3 Dynamic resistance

Since the current at resonance is in phase with the voltage, the impedance of the network acts as a resistance. This resistance is known as the **dynamic resistance**, \( R_D \). Impedance at resonance, \( R_D = V/I_r \), where \( I_r \) is the current at resonance.

\[
I_r = VY_r = V \left( \frac{R}{R^2 + \omega_0^2 L^2} \right)
\]

from equation (29.1) with the j terms equal to zero.

Hence \( R_D = \frac{V}{I_r} = \frac{V}{VR/(R^2 + \omega_0^2 L^2)} = \frac{R^2 + \omega_0^2 L^2}{R} \)

\[
= \frac{R^2 + L^2(1/LC) - (R^2/L^2)}{R} \quad \text{from equation } (29.2)
\]

\[
= \frac{R^2 + (L/C) - R^2}{R} = \frac{L/C - R}{R} = \frac{L}{CR}
\]

Hence **dynamic resistance**, \( R_D = \frac{L}{CR} \) \( \tag{29.4} \)

### 29.4 The \( LR-CR \) parallel network

A more general network comprising a coil of inductance \( L \) and resistance \( R_L \) in parallel with a capacitance \( C \) and resistance \( R_C \) in series is shown in Figure 29.4.
Admittance of inductive branch,

\[ Y_L = \frac{1}{R_L + jX_L} = \frac{R_L - jX_L}{R_L^2 + X_L^2} = \frac{R_L}{R_L^2 + X_L^2} - \frac{jX_L}{R_L^2 + X_L^2} \]

Admittance of capacitive branch,

\[ Y_C = \frac{1}{R_C - jX_C} = \frac{R_C + jX_C}{R_C^2 + X_C^2} = \frac{R_C}{R_C^2 + X_C^2} + \frac{jX_C}{R_C^2 + X_C^2} \]

Total network admittance,

\[ Y = Y_L + Y_C = \frac{R_L}{R_L^2 + jX_L^2} - \frac{jX_L}{R_L^2 + X_L^2} + \frac{R_C}{R_C^2 + jX_C^2} + \frac{jX_C}{R_C^2 + X_C^2} \]

At resonance the admittance is a minimum, i.e., when the imaginary part of \( Y \) is zero. Hence, at resonance,

\[ \frac{-X_L}{R_L^2 + X_L^2} + \frac{X_C}{R_C^2 + X_C^2} = 0 \]

i.e.,

\[ \frac{\omega_L}{R_L^2 + \omega^2 L^2} = \frac{1/(\omega_C)}{R_C^2 + 1/\omega_C^2} \]  \( (29.5) \)

Rearranging gives:

\[ \omega_L \left( R_C^2 + \frac{1}{\omega_C^2} \right) = \frac{1}{\omega_C} (R_L^2 + \omega_L^2 L^2) \]

\[ \omega_L R_C^2 + \frac{L}{\omega_C^2} = \frac{R_L^2}{\omega_C} + \omega_L L^2 \]

Multiplying throughout by \( \omega_C^2 \) gives:

\[ \omega_C^2 C^2 LR_C^2 + L = R_L^2 C + \omega_L^2 L^2 C \]

\[ \omega_C^2 (C^2 LR_C^2 - L^2 C) = R_L^2 C - L \]

\[ \omega_C^2 CL(CR_C^2 - L) = R_L^2 C - L \]

Hence \( \omega_C^2 = \frac{(CR_C^2 - L)}{LC(CR_C^2 - L)} \)

i.e., \( \omega_C = \frac{1}{\sqrt{(LC)}} \sqrt{\left( \frac{R_L^2 - (L/C)}{R_C^2 - (L/C)} \right)} \)

Hence resonant frequency, \( f_r = \frac{1}{2\pi\sqrt{(LC)}} \sqrt{\left( \frac{R_L^2 - (L/C)}{R_C^2 - (L/C)} \right)} \)  \( (29.6) \)
It is clear from equation (29.5) that parallel resonance may be achieved in such a circuit in several ways—by varying either the frequency \( f \), the inductance \( L \), the capacitance \( C \), the resistance \( R_L \) or the resistance \( R_C \).

29.5 Q-factor in a parallel network

The Q-factor in the series \( R–L–C \) circuit is a measure of the voltage magnification. In a parallel circuit, currents higher than the supply current can circulate within the parallel branches of a parallel resonant network, the current leaving the capacitor and establishing the magnetic field of the inductance, this then collapsing and recharging the capacitor, and so on. The Q-factor of a parallel resonant circuit is the ratio of the current circulating in the parallel branches of the circuit to the supply current, i.e. in a parallel circuit, Q-factor is a measure of the current magnification.

Circulating currents may be several hundreds of times greater than the supply current at resonance. For the parallel network of Figure 29.5, the Q-factor at resonance is given by:

\[
Q_r = \frac{\text{circulating current at resonance}}{\text{current at resonance}} = \frac{\text{capacitor current at resonance}}{I_r} = \frac{I_C}{I_r}
\]

Current in capacitor, \( I_C = V/X_C = V\omega_r C \)

Current at resonance, \( I_r = \frac{V}{R_D} = \frac{V}{L/CR} = \frac{VCR}{L} \)

Hence \( Q_r = \frac{I_C}{I_r} = \frac{V\omega_r C}{VCR/L} \) i.e., \( Q_r = \frac{\omega_r L}{R} \)

the same expression as for series resonance.

The difference between the resonant frequency of a series circuit and that of a parallel circuit can be quite small. The resonant frequency of a coil in parallel with a capacitor is shown in Equation (29.3); however, around the closed loop comprising the coil and capacitor the energy would naturally resonate at a frequency given by that for a series \( R–L–C \) circuit, as shown in Chapter 28. This latter frequency is termed the natural frequency, \( f_n \), and the frequency of resonance seen at the terminals of Figure 29.5 is often called the forced resonant frequency, \( f_r \). (For a series circuit, the forced and natural frequencies coincide.)

From the coil-capacitor loop of Figure 29.5, \( f_n = \frac{1}{2\pi\sqrt{(LC)}} \)

and the forced resonant frequency, \( f_r = \frac{1}{2\pi} \sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \)
Thus \( f_r \neq f_n \)
\[
\frac{f_r}{f_n} = 1 \quad \frac{2\pi}{\sqrt{(1 - \frac{R^2}{L^2})}} = \sqrt{1 - \frac{R^2}{L^2}}
\]
\[
= \sqrt{\left(1 - \frac{R^2}{L^2}\right)} \quad \sqrt{(LC)} = \sqrt{\left(\frac{LC}{LC} - \frac{LCR^2}{L^2}\right)} = \sqrt{\left(1 - \frac{R^2C}{L}\right)}
\]

From Chapter 28, \( Q = \frac{1}{R} \sqrt{\frac{L}{C}} \) from which

\[
Q^2 = \frac{1}{R^2} \left(\frac{L}{C}\right) \text{ or } \frac{R^2C}{L} = \frac{1}{Q^2}
\]

Hence \( f_r = f_n \sqrt{\left(1 - \frac{R^2C}{L}\right)} = \sqrt{\left(1 - \frac{1}{Q^2}\right)} \)

\[\text{i.e.,} \quad f_r = f_n \sqrt{\left(1 - \frac{1}{Q^2}\right)}\]

Thus it is seen that even with small values of \( Q \) the difference between \( f_r \) and \( f_n \) tends to be very small. A high value of \( Q \) makes the parallel resonant frequency tend to the same value as that of the series resonant frequency.

The expressions already obtained in Chapter 28 for bandwidth and resonant frequency, also apply to parallel circuits,

\[\text{i.e.,} \quad Q_r = f_r / (f_2 - f_1) \quad (29.7)\]

and \[f_r = \sqrt{(f_1f_2)} \quad (29.8)\]

The overall Q-factor \( Q_T \) of two parallel components having different Q-factors is given by:

\[Q_T = \frac{Q_1Q_c}{Q_1 + Q_c} \quad (29.9)\]

as for the series circuit.

By similar reasoning to that of the series \( R-L-C \) circuit it may be shown that at the half-power frequencies the admittance is \( \sqrt{2} \) times its minimum value at resonance and, since \( Z = 1/Y \), the value of impedance
at the half-power frequencies is $1/\sqrt{2}$ or 0.707 times its maximum value at resonance.

By similar analysis to that given in Chapter 28, it may be shown that for a parallel network:

$$\frac{Y}{Y_r} = \frac{R_D}{Z} = 1 + j2Q$$  \hspace{1cm} (29.10)

where $Y$ is the circuit admittance, $Y_r$ is the admittance at resonance, $Z$ is the network impedance and $R_D$ is the dynamic resistance (i.e., the impedance at resonance) and $\delta$ is the fractional deviation from the resonant frequency.

Problem 1. A coil of inductance 5 mH and resistance 10 $\Omega$ is connected in parallel with a 250 nF capacitor across a 50 V variable-frequency supply. Determine (a) the resonant frequency, (b) the dynamic resistance, (c) the current at resonance, and (d) the circuit Q-factor at resonance.

(a) Resonance frequency

$$f_r = \frac{1}{2\pi} \sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \text{ from equation (29.3),}$$

$$= \frac{1}{2\pi} \sqrt{\left(\frac{1}{5 \times 10^{-3} \times 250 \times 10^{-9}} - \frac{10^2}{(5 \times 10^{-3})^2}\right)}$$

$$= \frac{1}{2\pi} \sqrt{(800 \times 10^6 - 4 \times 10^6)} = \frac{1}{2\pi} \sqrt{796 \times 10^6} = 4490 \text{ Hz}$$

(b) From equation (29.4), dynamic resistance,

$$R_D = \frac{L}{CR} = \frac{5 \times 10^{-3}}{(250 \times 10^{-9})(10)} = 2000 \text{ $\Omega$}$$

(c) Current at resonance, $I_r = \frac{V}{R_D} = \frac{50}{2000} = 25 \text{ mA}$

(d) Q-factor at resonance, $Q_r = \frac{\omega_0 L}{R} = \frac{(2\pi \times 4490)(5 \times 10^{-3})}{10} = 14.1$

Problem 2. In the parallel network of Figure 29.6, inductance, $L = 100$ mH and capacitance, $C = 40$ $\mu$F. Determine the resonant frequency for the network if (a) $R_L = 0$ and (b) $R_L = 30$ $\Omega$
Total circuit admittance,

\[
Y = \frac{1}{R_L + jX_L} + \frac{1}{-jX_C} = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}
\]

\[
= \frac{R_L}{R_L^2 + X_L^2} - \frac{jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}
\]

The network is at resonance when the admittance is at a minimum value, i.e., when the imaginary part is zero. Hence, at resonance,

\[
\frac{-X_L}{R_L^2 + X_L^2} + \frac{1}{X_C} = 0 \quad \text{or} \quad \omega_r C = \frac{\omega_r L}{R_L^2 + \omega_r^2 L^2}
\]

(a) When \(R_L = 0\), \(\omega_r C = \frac{\omega_r L}{\omega_r^2 L^2}\) from which, \(\omega_r^2 = \frac{1}{LC}\) and \(\omega_r = \sqrt{(LC)}\)

Hence resonant frequency,

\[
f_r = \frac{1}{2\pi \sqrt{(LC)}} = \frac{1}{2\pi \sqrt{(100 \times 10^{-3} \times 40 \times 10^{-6})}} = 79.6 \text{ Hz}
\]

(b) When \(R_L = 30\Omega\), \(\omega_r C = \frac{\omega_r L}{30^2 + \omega_r^2 L^2}\) from equation (29.11) above

from which,

\[
30^2 + \omega_r^2 L^2 = \frac{L}{C}
\]

i.e.,

\[
\omega_r^2 (100 \times 10^{-3})^2 = \frac{100 \times 10^{-3}}{40 \times 10^{-6}} = 900
\]

i.e., \(\omega_r^2(0.01) = 2500 - 900 = 1600\)

Thus, \(\omega_r^2 = 1600/0.01 = 160{,}000\) and \(\omega_r = \sqrt{160{,}000} = 400 \text{ rad/s}\)

Hence resonant frequency, \(f_r = \frac{400}{2\pi} = 63.7 \text{ Hz}\)

[Alternatively, from equation (29.3),

\[
f_r = \frac{1}{2\pi} \sqrt{\left( \frac{L}{C} - \frac{R^2}{L^2} \right) - \frac{30^2}{(100 \times 10^{-3})^2}}
\]

\[
= \frac{1}{2\pi} \sqrt{\frac{1}{(100 \times 10^{-3}(40 \times 10^{-6}) - \frac{30^2}{(100 \times 10^{-3})^2}}}
\]

\[
= \frac{1}{2\pi} \sqrt{(250{,}000 - 90{,}000)} = \frac{1}{2\pi} \sqrt{160{,}000} = \frac{1}{2\pi}(400) = 63.7 \text{ Hz}\]
Hence, as the resistance of a coil increases, the resonant frequency decreases in the circuit of Figure 29.6.

Problem 3. A coil of inductance 120 mH and resistance 150 Ω is connected in parallel with a variable capacitor across a 20 V, 4 kHz supply. Determine for the condition when the supply current is a minimum, (a) the capacitance of the capacitor, (b) the dynamic resistance, (c) the supply current, (d) the Q-factor, (e) the bandwidth, (f) the upper and lower −3 dB frequencies, and (g) the value of the circuit impedance at the −3 dB frequencies.

(a) The supply current is a minimum when the parallel network is at resonance.

Resonant frequency, \( f_r = \frac{1}{2\pi} \sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \) from equation (29.3),

from which, \( (2\pi f_r)^2 = \frac{1}{LC} - \frac{R^2}{L^2} \)

Hence \( \frac{1}{LC} = (2\pi f_r)^2 + \frac{R^2}{L^2} \) and

capacitance \( C = \frac{1}{L[(2\pi f_r)^2 + (R^2/L^2)]} \)

\[
C = \frac{1}{120 \times 10^{-3}[(2\pi4000)^2 + (150^2/(120 \times 10^{-3})^2)]}
\]

\[
= \frac{1}{0.12(631.65 \times 10^6 + 1.5625 \times 10^6)}
\]

\[= 0.01316 \ \mu F \ or \ 13.16 \ nF\]

(b) Dynamic resistance, \( R_D = \frac{L}{CR} = \frac{120 \times 10^{-3}}{(13.16 \times 10^{-9})(150)} \)

\[= 60.79 \ \text{kΩ}\]

(c) Supply current at resonance,

\[ I_r = \frac{V}{R_D} = \frac{20}{60.79 \times 10^{-3}} = 0.329 \ \text{mA} \ or \ 329 \ \mu A\]

(d) Q-factor at resonance, \( Q_r = \frac{\omega_r L}{R} = \frac{(2\pi4000)(120 \times 10^{-3})}{150} \)

\[= 20.11\]

[Note that the expressions \( Q_r = \frac{1}{\omega_r CR} \) or \( Q_r = \frac{1}{R} \sqrt{\left(\frac{L}{C}\right)} \)]
used for the $R-L-C$ series circuit may also be used in parallel circuits when the resistance of the coil is much smaller than the inductive reactance of the coil.

In this case $R = 150$ Ω and $X_L = 2\pi(4000)(120 \times 10^{-3}) = 3016$ Ω.

Hence, alternatively,

$$Q_r = \frac{1}{\omega C R} = \frac{1}{(2\pi 4000)(13.16 \times 10^{-9})(150)} = 20.16$$

or $Q_r = \frac{1}{R} \sqrt{\left(\frac{L}{C}\right)} = \frac{1}{150} \sqrt{\left(\frac{120 \times 10^{-3}}{13.16 \times 10^{-9}}\right)} = 20.13$

(e) If the lower and upper $-3$ dB frequencies are $f_1$ and $f_2$ respectively then the bandwidth is $(f_2 - f_1)$. Q-factor at resonance is given by $Q_r = f_r/(f_2 - f_1)$. From which, bandwidth,

$$(f_2 - f_1) = \frac{f_r}{Q_r} = \frac{4000}{20.11} = 199 \text{ Hz}$$

(f) Resonant frequency, $f_r = \sqrt{(f_1 f_2)}$, from which

$$f_1 f_2 = f_r^2 = (4000)^2 = 16 \times 10^6 \quad \text{(29.12)}$$

Also, from part (e), $f_2 - f_1 = 199 \quad \text{(29.13)}$

From equation (29.12),

$$f_1 = \frac{16 \times 10^6}{f_2}$$

Substituting in equation (29.13) gives:

$$f_2 - \frac{16 \times 10^6}{f_2} = 199$$

i.e.,

$$f_2^2 - 16 \times 10^6 = 199 f_2$$

from which,

$$f_2^2 - 199 f_2 - 16 \times 10^6 = 0.$$ 

Solving this quadratic equation gives:

$$f_2 = \frac{199 \pm \sqrt{[(199)^2 - 4(-16 \times 10^6)]}}{2} = \frac{199 \pm 8002.5}{2}$$

i.e., the upper $3$ dB frequency, $f_2 = 4100$ Hz (neglecting the negative answer).

From equation (29.12),

the lower $-3$ dB frequency, $f_1 = \frac{10 \times 10^6}{f_2} = \frac{16 \times 10^6}{4100}$

$$= 3900 \text{ Hz}$$
(Note that $f_1$ and $f_2$ are equally displaced about the resonant frequency, $f_r$, as they always will be when $Q$ is greater than about 10—just as for a series circuit)

(g) The value of the circuit impedance, $Z$, at the $-3$ dB frequencies is given by

$$Z = \frac{1}{\sqrt{2}} Z_r$$

where $Z_r$ is the impedance at resonance.

The impedance at resonance $Z_r = R_D$, the dynamic resistance.

Hence **impedance at the $-3$ dB frequencies** = \( \frac{1}{\sqrt{2}} (60.79 \times 10^3) \)

= **42.99 kΩ**

Figure 29.7 shows impedance plotted against frequency for the circuit in the region of the resonant frequency.

**Problem 4.** A two-branch parallel network is shown in Figure 29.8. Determine the resonant frequency of the network.

From equation (29.6),

resonant frequency, $f_r = \frac{1}{2 \pi \sqrt{LC}} \left( \frac{R_L}{R_L - (L/C)} \right)$

where $R_L = 5 \, \Omega$, $R_C = 3 \, \Omega$, $L = 2 \, \text{mH}$ and $C = 25 \, \mu\text{F}$. Thus

$$f_r = \frac{1}{2 \pi \sqrt{(2 \times 10^{-3})(25 \times 10^{-6})}} \sqrt{\left( \frac{5^2 - ((2 \times 10^{-3})/(25 \times 10^{-6}))}{3^2 - ((2 \times 10^{-3})/(25 \times 10^{-6}))} \right)}$$

$$= \frac{1}{2 \pi \sqrt{(5 \times 10^{-8})}} \sqrt{\left( \frac{25 - 80}{9 - 80} \right)}$$

$$= \frac{10^4}{2 \pi \sqrt{5}} \sqrt{\left( \frac{-55}{-71} \right)} = 626.5 \, \text{Hz}$$

**Problem 5.** Determine for the parallel network shown in Figure 29.9 the values of inductance $L$ for which the network is resonant at a frequency of 1 kHz.

Figure 29.7

![Impedance plot](image)

Figure 29.8

![Parallel network](image)

Figure 29.9

![Parallel network with 2H resistor](image)
The total network admittance, \( Y \), is given by

\[
Y = \frac{1}{3 + jX_L} + \frac{1}{4 - j10} = \frac{3 - jX_L}{3^2 + X_L^2} + \frac{4 + j10}{4^2 + 10^2}
\]

\[
= \frac{3}{3^2 + X_L^2} - \frac{jX_L}{3^2 + X_L^2} + \frac{4}{116} + \frac{j10}{116}
\]

\[
= \left( \frac{3}{3^2 + X_L^2} + \frac{4}{116} \right) + j \left( \frac{10}{116} - \frac{X_L}{3^2 + X_L^2} \right)
\]

Resonance occurs when the admittance is a minimum, i.e., when the imaginary part of \( Y \) is zero. Hence, at resonance,

\[
\frac{10}{116} - \frac{X_L}{3^2 + X_L^2} = 0 \quad \text{i.e.,} \quad \frac{10}{116} = \frac{X_L}{3^2 + X_L^2}
\]

Therefore \( 10(9 + X_L^2) = 116 \, X_L \), i.e., \( 10 \, X_L^2 - 116 \, X_L + 90 = 0 \)

from which, \( X_L^2 - 11.6 \, X_L + 9 = 0 \)

Solving the quadratic equation gives:

\[
X_L = \frac{11.6 \pm \sqrt{\left(-11.6\right)^2 - 4(9)}}{2} = \frac{11.6 \pm 9.93}{2}
\]

i.e., \( X_L = 10.765 \, \Omega \) or 0.835 \, \Omega. Hence \( 10.765 = 2\pi f_1 L_1 \), from which,

inductance \( L_1 = \frac{10.765}{2\pi(1000)} = 1.71 \, \text{mH} \)

and 0.835 = \( 2\pi f_1 L_2 \) from which,

inductance, \( L_2 = \frac{0.835}{2\pi(1000)} = 0.13 \, \text{mH} \)

Thus the conditions for the circuit of Figure 29.9 to be resonant are that inductance \( L \) is either 1.71 mH or 0.13 mH

Problem 6. A capacitor having a Q-factor of 300 is connected in parallel with a coil having a Q-factor of 60. Determine the overall Q-factor of the parallel combination.

From equation (29.9), the overall Q-factor is given by:

\[
Q_T = \frac{Q_L Q_C}{Q_L + Q_C} = \frac{(60)(300)}{60 + 300} = \frac{18000}{360} = 50
\]
Problem 7. In an LR–C network, the capacitance is 10.61 nF, the bandwidth is 500 Hz and the resonant frequency is 150 kHz. Determine for the circuit (a) the Q-factor, (b) the dynamic resistance, and (c) the magnitude of the impedance when the supply frequency is 0.4% greater than the tuned frequency.

(a) From equation (29.7), \[ Q = \frac{f_r}{f_2 - f_1} = \frac{150 \times 10^3}{500} = 300 \]

(b) From equation (29.4), dynamic resistance, \[ R_D = \frac{L}{CR} \]

Also, in an LR–C network, \[ Q = \frac{\omega LC}{R} \] from which, \[ R = \frac{\omega L}{Q} \]

Hence, \[ R_D = \frac{L}{CR} = \frac{L}{C \left( \frac{\omega L}{Q} \right)} = \frac{LQ}{\omega C} = \frac{Q}{\omega C} \]

\[ = \frac{300}{(2\pi 150 \times 10^3)(10.61 \times 10^{-9})} = 30 \text{ k}\Omega \]

(c) From equation (29.10), \[ \frac{R_D}{Z} = 1 + j2\delta Q \] from which, \[ Z = \frac{R_D}{1 + j2\delta Q} \]

\[ \delta = 0.4\% = 0.004 \text{ hence } Z = \frac{30 \times 10^3}{1 + j2(0.004)(300)} \]

\[ = \frac{30 \times 10^3}{1 + j2.4} = \frac{30 \times 10^3}{2.6 \times 67.38} \]

\[ = 11.54 \angle -67.38^\circ \text{ k}\Omega \]

Hence the magnitude of the impedance when the frequency is 0.4% greater than the tuned frequency is 11.54 k\Omega.

Further problems on parallel resonance may be found in the Section 29.6 following, problems 1 to 14.

29.6 Further problems on parallel resonance and Q-factor

1. A coil of resistance 20 \( \Omega \) and inductance 100 mH is connected in parallel with a 50 \( \mu \text{F} \) capacitor across a 30 V variable-frequency supply. Determine (a) the resonant frequency of the circuit, (b) the dynamic resistance, (c) the current at resonance, and (d) the circuit Q-factor at resonance.  
   [(a) 63.66 Hz (b) 100 \( \Omega \) (c) 0.30 A (d) 2]

2. A 25 V, 2.5 kHz supply is connected to a network comprising a variable capacitor in parallel with a coil of resistance 250 \( \Omega \) and inductance 80 mH. Determine for the condition when the supply
current is a minimum (a) the capacitance of the capacitor, (b) the
dynamic resistance, (c) the supply current, (d) the Q-factor, (e) the
bandwidth, (f) the upper and lower half-power frequencies and
(g) the value of the circuit impedance at the \(-3\) dB frequencies.

\[(a) \ 48.73 \text{ nF} \quad (b) \ 6.57 \text{ k\Omega} \quad (c) \ 3.81 \text{ mA} \quad (d) \ 5.03 \quad (e) \ 497.3 \text{ Hz} \quad (f) \ 2761 \text{ Hz} \quad (g) \ 4.64 \text{ k\Omega}\]

3 A 0.1 \(\mu\text{F}\) capacitor and a pure inductance of 0.02 H are connected in parallel across a 12 V variable-frequency supply. Determine (a) the resonant frequency of the circuit, and (b) the current circulating in the capacitance and inductance at resonance.

\[(a) \ 3.56 \text{ kHz} \quad (b) \ 26.84 \text{ mA}\]

4 A coil of resistance 300 \(\Omega\) and inductance 100 mH and a 4000 pF capacitor are connected (i) in series and (ii) in parallel. Find for each connection (a) the resonant frequency, (b) the Q-factor, and (c) the impedance at resonance.

\[(i) \ (a) \ 7958 \text{ Hz} \quad (b) \ 16.67 \quad (c) \ 300 \text{ \(\Omega\)} \quad (ii) \ (a) \ 7943 \text{ Hz} \quad (b) \ 16.64 \quad (c) \ 83.33 \text{ k\Omega}\]

5 A network comprises a coil of resistance 100 \(\Omega\) and inductance 0.8 H and a capacitor having capacitance 30 \(\mu\text{F}\). Determine the resonant frequency of the network when the capacitor is connected (a) in series with, and (b) in parallel with the coil.

\[(a) \ 32.5 \text{ Hz} \quad (b) \ 25.7 \text{ Hz}\]

6 Determine the value of capacitor \(C\) shown in Figure 29.10 for which the resonant frequency of the network is 1 kHz.

\[2.30 \mu\text{F}\]

7 In the parallel network shown in Figure 29.11, inductance \(L\) is 40 mH and capacitance \(C\) is 5 \(\mu\text{F}\). Determine the resonant frequency of the circuit if (a) \(R_L = 0\) and (b) \(R_L = 40 \text{ \(\Omega\)}\).

\[(a) \ 355.9 \text{ Hz} \quad (b) \ 318.3 \text{ Hz}\]

8 A capacitor of reactance 5 \(\Omega\) is connected in series with a 10 \(\Omega\) resistor. The whole circuit is then connected in parallel with a coil of inductive reactance 20 \(\Omega\) and a variable resistor. Determine the value of this resistance for which the parallel network is resonant.

\[10 \text{ \(\Omega\)}\]

9 Determine, for the parallel network shown in Figure 29.12, the values of inductance \(L\) for which the circuit is resonant at a frequency of 600 Hz.

\[2.50 \text{ mH or 0.45 mH}\]

10 Find the resonant frequency of the two-branch parallel network shown in Figure 29.13.

\[667 \text{ Hz}\]

11 Determine the value of the variable resistance \(R\) in Figure 29.14 for which the parallel network is resonant.

\[11.87 \text{ \(\Omega\)}\]

12 For the parallel network shown in Figure 29.15, determine the resonant frequency. Find also the value of resistance to be connected in series with the 10 \(\mu\text{F}\) capacitor to change the resonant frequency to 1 kHz.

\[928 \text{ Hz; 5.27 \(\Omega\)}\]
13 Determine the overall Q-factor of a parallel arrangement consisting of a capacitor having a Q-factor of 410 and an inductor having a Q-factor of 90. [73.8]

14 The value of capacitance in an $LR-C$ parallel network is 49.74 nF. If the resonant frequency of the circuit is 200 kHz and the bandwidth is 800 Hz, determine for the network (a) the Q-factor, (b) the dynamic resistance, and (c) the magnitude of the impedance when the supply frequency is 0.5% smaller than the tuned frequency.

[(a) 250 (b) 4 k$\Omega$ (c) 1.486 k$\Omega$]