Two-port network

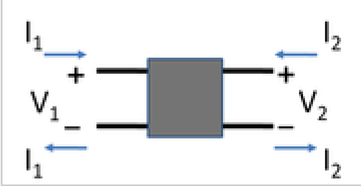

Figure 1: Example two-port network with symbol definitions. Notice the **port condition** is satisfied: the same current flows into each port as leaves that port.

A **two-port network** (or **four-terminal network** or **quadripole**) is an electrical circuit or device with two *pairs* of terminals (i.e., the circuit connects two dipoles). Two terminals constitute a **port** if they satisfy the essential requirement known as the **port condition**: the same current must enter and leave a port.[1][2] Examples include small-signal models for transistors (such as the hybrid-pi model), filters and matching networks. The analysis of passive two-port networks is an outgrowth of reciprocity theorems first derived by Lorentz[3].

A two-port network makes possible the isolation of either a complete circuit or part of it and replacing it by its characteristic parameters. Once this is done, the isolated part of the circuit becomes a "black box" with a set of distinctive properties, enabling us to abstract away its specific physical buildup, thus simplifying analysis. Any linear circuit with four terminals can be transformed into a two-port network provided that it does not contain an independent source and satisfies the port conditions.

The parameters used to describe a two-port network are *z*, *y*, *h*, *g*, and *T*. They are usually expressed in matrix notation, and they establish relations between the variables

$V_1 \stackrel{\text{def}}{=}$ Input voltage
$V_2 \stackrel{\text{def}}{=}$ Output voltage
$I_1 \stackrel{\text{def}}{=}$ Input current
$I_2 \stackrel{\text{def}}{=}$ Output current

which are shown in Figure 1. These current and voltage variables are most useful at low-to-moderate frequencies. At high frequencies (e.g., microwave frequencies), the use of power and energy variables is more appropriate, and the two-port current–voltage approach that is discussed here is replaced by an approach based upon scattering parameters.

Though some authors use the terms *two-port network* and *four-terminal network* interchangeably, the latter represents a more general concept. Not all four-terminal networks are two-port networks. A pair of terminals can be called a *port* only if the current entering one is equal to the current leaving the other; this definition is called the *port condition*. Only those four-terminal networks consisting of two *ports* can be called two-port networks.[1][2]

Impedance parameters (z-parameters)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

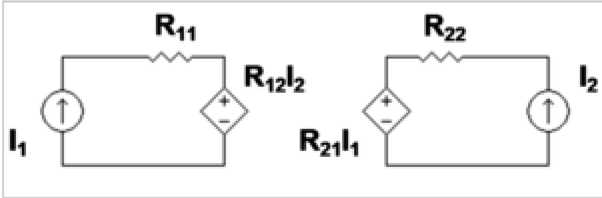

Figure 2: z-equivalent two port showing independent variables I_1 and I_2. Although resistors are shown, general impedances can be used instead.

where

$$z_{11} \stackrel{\text{def}}{=} \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{12} \stackrel{\text{def}}{=} \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$z_{21} \stackrel{\text{def}}{=} \left.\frac{V_2}{I_1}\right|_{I_2=0} \qquad z_{22} \stackrel{\text{def}}{=} \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

Notice that all the z-parameters have dimensions of ohms.

Example: bipolar current mirror with emitter degeneration

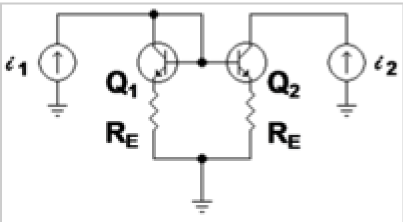

Figure 3: Bipolar current mirror: i_1 is the *reference current* and i_2 is the *output current*; lower case symbols indicate these are *total* currents that include the DC components

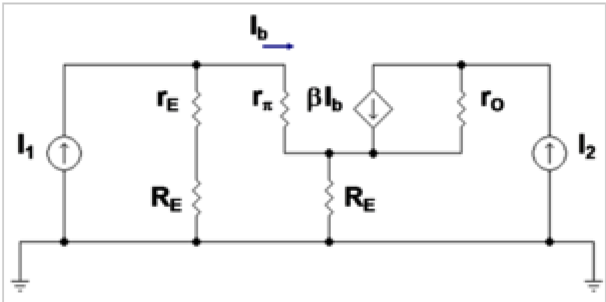

Figure 4: Small-signal bipolar current mirror: I_1 is the amplitude of the small-signal *reference current* and I_2 is the amplitude of the small-signal *output current*

Figure 3 shows a bipolar current mirror with emitter resistors to increase its output resistance.[nb 1] Transistor Q_1 is *diode connected*, which is to say its collector-base voltage is zero. Figure 4 shows the small-signal circuit equivalent to Figure 3. Transistor Q_1 is represented by its emitter resistance $r_E \approx V_T / I_E$ (V_T = thermal voltage, I_E = Q-point emitter current), a simplification made possible because the dependent current source in the hybrid-pi model for Q_1 draws the same current as a resistor 1 / g_m connected across r_π. The second transistor Q_2 is represented by its hybrid-pi model. Table 1 below shows the z-parameter expressions that make the z-equivalent circuit of Figure 2 electrically equivalent to the small-signal circuit of Figure 4.

Table 1	Expression	Approximation
$R_{21} = \left.\frac{V_2}{I_1}\right\|_{I_2=0}$	$-(\beta r_O - R_E)\frac{r_E+R_E}{r_\pi+r_E+2R_E}$	$-\beta r_o \frac{r_E+R_E}{r_\pi+2R_E}$
$R_{11} = \left.\frac{V_1}{I_1}\right\|_{I_2=0}$	$(r_E+R_E)\\|(r_\pi+R_E)$	
$R_{22} = \left.\frac{V_2}{I_2}\right\|_{I_1=0}$	$(1+\beta\frac{R_E}{r_\pi+r_E+2R_E})r_O + \frac{r_\pi+r_E+R_E}{r_\pi+r_E+2R_E}R_E$	$(1+\beta\frac{R_E}{r_\pi+2R_E})r_O$

$R_{12} = \left.\frac{V_1}{I_2}\right\|_{I_1=0}$	$R_E \frac{r_E+R_E}{r_\pi+r_E+2R_E}$	$R_E \frac{r_E+R_E}{r_\pi+2R_E}$

The negative feedback introduced by resistors R_E can be seen in these parameters. For example, when used as an active load in a differential amplifier, $I_1 \approx -I_2$, making the output impedance of the mirror approximately R_{22} -$R_{21} \approx 2\ \beta\ r_O R_E /(r_\pi+2R_E)$ compared to only r_O without feedback (that is with R_E = 0 Ω) . At the same time, the impedance on the reference side of the mirror is approximately $R_{11} - R_{12} \approx \frac{r_\pi}{r_\pi + 2R_E}(r_E + R_E)$, only a moderate value, but still larger than r_E with no feedback. In the differential amplifier application, a large output resistance increases the difference-mode gain, a good thing, and a small mirror input resistance is desirable to avoid Miller effect.

Admittance parameters (y-parameters)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

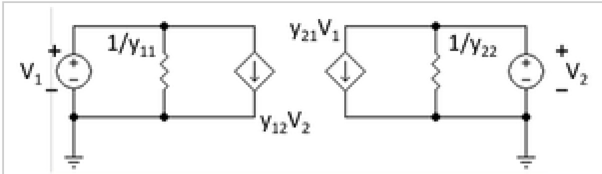

Figure 5: Y-equivalent two port showing independent variables V_1 and V_2. Although resistors are shown, general admittances can be used instead.

where

$$y_{11} \overset{\text{def}}{=} \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad y_{12} \overset{\text{def}}{=} \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

$$y_{21} \overset{\text{def}}{=} \left.\frac{I_2}{V_1}\right|_{V_2=0} \qquad y_{22} \overset{\text{def}}{=} \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

The network is said to be reciprocal if $y_{12} = y_{21}$. Notice that all the Y-parameters have dimensions of siemens.

Hybrid parameters (h-parameters)

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

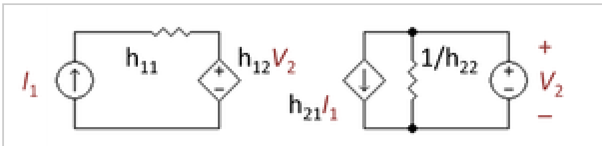

Figure 6: H-equivalent two-port showing independent variables I_1 and V_2; h_{22} is reciprocated to make a resistor

where

$$h_{11} \overset{\text{def}}{=} \left.\frac{V_1}{I_1}\right|_{V_2=0} \qquad h_{12} \overset{\text{def}}{=} \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} \overset{\text{def}}{=} \left.\frac{I_2}{I_1}\right|_{V_2=0} \qquad h_{22} \overset{\text{def}}{=} \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

Often this circuit is selected when a current amplifier is wanted at the output. The resistors shown in the diagram can be general impedances instead.

Notice that off-diagonal h-parameters are dimensionless, while diagonal members have dimensions the reciprocal of one another.

Example: common-base amplifier

Note: Tabulated formulas in Table 2 make the h-equivalent circuit of the transistor from Figure 6 agree with its small-signal low-frequency hybrid-pi model in Figure 7. Notation: r_π = base resistance of transistor, r_O = output resistance, and g_m = transconductance. The negative sign for h_{21} reflects the convention that I_1, I_2 are positive when directed *into* the two-port. A non-zero value for h_{12} means the output voltage affects the input voltage, that is, this amplifier is **bilateral**. If h_{12} = 0, the amplifier is **unilateral**.

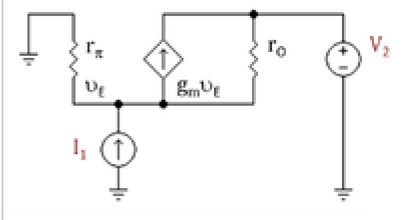

Figure 7: Common-base amplifier with AC current source I_1 as signal input and unspecified load supporting voltage V_2 and a dependent current I_2.

Table 2	Expression	Approximation
$h_{21} = \left.\frac{I_2}{I_1}\right\|_{V_2=0}$	$-\frac{\frac{\beta}{\beta+1}r_O+r_E}{r_O+r_E}$	$-\frac{\beta}{\beta+1}$
$h_{11} = \left.\frac{V_1}{I_1}\right\|_{V_2=0}$	$r_E \\| r_O$	r_E
$h_{22} = \left.\frac{I_2}{V_2}\right\|_{I_1=0}$	$\frac{1}{(\beta+1)(r_O+r_E)}$	$\frac{1}{(\beta+1)r_O}$
$h_{12} = \left.\frac{V_1}{V_2}\right\|_{I_1=0}$	$\frac{r_E}{r_E+r_O}$	$\frac{r_E}{r_O} \ll 1$

Inverse hybrid parameters (g-parameters)

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

where

$$g_{11} \overset{\text{def}}{=} \left.\frac{I_1}{V_1}\right|_{I_2=0} \qquad g_{12} \overset{\text{def}}{=} \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

$$g_{21} \overset{\text{def}}{=} \left.\frac{V_2}{V_1}\right|_{I_2=0} \qquad g_{22} \overset{\text{def}}{=} \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

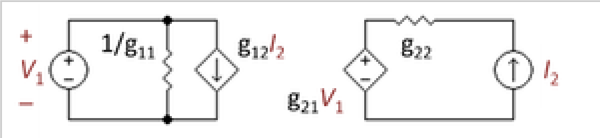

Figure 8: G-equivalent two-port showing independent variables V_1 and I_2; g_{11} is reciprocated to make a resistor

Often this circuit is selected when a voltage amplifier is wanted at the output. Notice that off-diagonal g-parameters are dimensionless, while diagonal members have dimensions the reciprocal of one another. The resistors shown in the diagram can be general impedances instead.

Example: common-base amplifier

Note: Tabulated formulas in Table 3 make the g-equivalent circuit of the transistor from Figure 8 agree with its small-signal low-frequency hybrid-pi model in Figure 9. Notation: r_π = base resistance of transistor, r_O = output resistance, and g_m = transconductance. The negative sign for g_{12} reflects the convention that I_1, I_2

are positive when directed *into* the two-port. A non-zero value for g_{12} means the output current affects the input current, that is, this amplifier is **bilateral**. If $g_{12} = 0$, the amplifier is **unilateral**.

Table 3	Expression	Approximation
$g_{21} = \left.\frac{V_2}{V_1}\right\|_{I_2=0}$	$\frac{r_o}{r_\pi} + g_m r_O + 1$	$g_m r_O$
$g_{11} = \left.\frac{I_1}{V_1}\right\|_{I_2=0}$	$\frac{1}{r_\pi}$	$\frac{1}{r_\pi}$
$g_{22} = \left.\frac{V_2}{I_2}\right\|_{V_1=0}$	r_O	r_O
$g_{12} = \left.\frac{I_1}{I_2}\right\|_{V_1=0}$	$-\frac{\beta+1}{\beta}$	-1

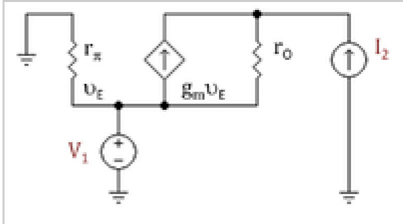

Figure 9: Common-base amplifier with AC voltage source V_1 as signal input and unspecified load delivering current I_2 at a dependent voltage V_2.

ABCD-parameters

The ABCD-parameters are known variously as chain, cascade, or transmission parameters.

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

where

$$A \stackrel{\text{def}}{=} \left.\frac{V_2}{V_1}\right|_{I_1=0} \qquad B \stackrel{\text{def}}{=} \left.\frac{V_2}{I_1}\right|_{V_1=0}$$

$$C \stackrel{\text{def}}{=} -\left.\frac{I_2}{V_1}\right|_{I_1=0} \qquad D \stackrel{\text{def}}{=} -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

Note that we have inserted negative signs in front of the fractions in the definitions of parameters *C* and *D*. The reason for adopting this convention (as opposed to the convention adopted above for the other sets of parameters) is that it allows us to represent the transmission matrix of cascades of two or more two-port networks as simple matrix multiplications of the matrices of the individual networks. This convention is equivalent to reversing the direction of I_2 so that it points in the same direction as the input current to the next stage in the cascaded network.

An ABCD matrix has been defined for Telephony four-wire Transmission Systems by P K Webb in British Post Office Research Department Report 630 in 1977.

Table of transmission parameters

The table below lists ABCD parameters for some simple network elements.

Element	Matrix	Remarks
Series resistor	$\begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix}$	*R* = resistance
Shunt resistor	$\begin{bmatrix} 1 & 0 \\ -1/R & 1 \end{bmatrix}$	*R* = resistance
Series conductor		*G* = conductance

	$\begin{bmatrix} 1 & -1/G \\ 0 & 1 \end{bmatrix}$	
Shunt conductor	$\begin{bmatrix} 1 & 0 \\ -G & 1 \end{bmatrix}$	G = conductance
Series inductor	$\begin{bmatrix} 1 & -Ls \\ 0 & 1 \end{bmatrix}$	L = inductance s = complex angular frequency
Shunt capacitor	$\begin{bmatrix} 1 & 0 \\ -Cs & 1 \end{bmatrix}$	C = capacitance s = complex angular frequency

Combinations of two-port networks

Series connection of two 2-port networks: **Z = Z1 + Z2**
Parallel connection of two 2-port networks: **Y = Y1 + Y2**

Example: Cascading two networks

Suppose we have a two-port network consisting of a series resistor R followed by a shunt capacitor C. We can model the entire network as a cascade of two simpler networks:

$$\mathbf{T}_1 = \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 \\ -Cs & 1 \end{bmatrix}$$

The transmission matrix for the entire network **T** is simply the matrix multiplication of the transmission matrices for the two network elements:

$$\begin{aligned} \mathbf{T} &= \mathbf{T}_2 \cdot \mathbf{T}_1 \\ &= \begin{bmatrix} 1 & 0 \\ -Cs & 1 \end{bmatrix} \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -R \\ -Cs & 1 + RCs \end{bmatrix} \end{aligned}$$

Thus:

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & -R \\ -Cs & 1 + RCs \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

Notes regarding definition of transmission parameters

1. It should be noted that all these examples are specific to the definition of transmission parameters given here. Other definitions exist in the literature, such as:
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

2. The format used above for cascading (ABCD) examples cause the "components" to be used backwards compared to standard electronics schematic conventions. This can be fixed by taking the transpose of the above formulas, or by making the V_1,I_1 the left hand side (dependent variables). Another advantage of the V_1,I_1 form is that the output can be terminated (via a transfer matrix representation of the load) and then I_2 can be set to zero; allowing the voltage transfer function, 1/A to be read directly.
3. In all cases the ABCD matrix terms and current definitions should allow cascading.

Networks with more than two ports

While two port networks are very common (e.g. amplifiers and filters), other electrical networks such as directional couplers and isolators have more than 2 ports. The following representations can be extended to networks with an arbitrary number of ports:

- Admittance (Y) parameters
- Impedance (Z) parameters
- Scattering (S) parameters

They are extended by adding appropriate terms to the matrix representing the other ports. So 3 port impedance parameters result in the following relationship:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

It should be noted that the following representations cannot be extended to more than two ports:

- Hybrid (h) parameters
- Inverse hybrid (g) parameters
- Transmission (ABCD) parameters
- Scattering transmission (T) parameters