Chapter 3

Series solutions of ODEs

*Pick a flower on earth and you move the farthest star.*

Paul A. M. Dirac (1902-1984)

3.1 Preliminary

3.1.1 General solution of second-order ODEs

Recall that the most general solution of a second-order linear ODE

\[
\frac{d^2 u}{dx^2} + p(x) \frac{du}{dx} + q(x)u = g(x),
\]

is

\[ u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p, \]

where \( c_1 \) and \( c_2 \) are constants, \( u_p \) is a particular solution for the nonhomogeneous ODE, and \( u_1 \) and \( u_2 \) are fundamental solutions that satisfy the homogeneous equation

\[
\frac{d^2 u}{dx^2} + p(x) \frac{du}{dx} + q(x)u = 0
\]

and the condition

\[ W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ \frac{du_1}{dx} & \frac{du_2}{dx} \end{vmatrix} \neq 0. \]

Here, \( W(u_1, u_2) \) is the Wronskian. By Abel’s theorem, it suffices to check the above condition at a single point in the open interval in which the solution
exists, since the Wronskian is either zero or different from zero. The above condition is equivalent to saying that \( u_1 \) and \( u_2 \) are linearly independent:

\[
\alpha u_1 + \beta u_2 = 0
\]

implies that \( \alpha = \beta = 0 \).\(^1\)

If one knows \( u_1 \), one can construct a second solution using the Ansatz \( u_2(x) = h(x)u_1(x) \). This method is called the method of variation of parameters.

In this chapter, we will discuss series solutions of the homogeneous equation

\[
\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0.
\]

### 3.2 Classification of singularities

In order to discuss types of singularities that appear in the coefficients of the ODE, we need to introduce the notion of a (real) analytic function.

**Definition of real analytic functions.**

A real function \( f(x) \) is real analytic at point \( x = x_0 \) if it can be represented by a power series in powers of \( x - x_0 \) with radius of convergence \( R > 0 \), i.e.,

\[
f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad |x - x_0| < R.
\]

\(^1\)Differentiating

\[
\alpha u_1 + \beta u_2 = 0
\]

implies

\[
\alpha u_1' + \beta u_2' = 0.
\]

We have a unique trivial solution \( \alpha = \beta = 0 \) when the determinant

\[
\begin{vmatrix}
    u_1 & u_2 \\
    \frac{du_1}{dx} & \frac{du_2}{dx}
\end{vmatrix} \neq 0.
\]
Examples.

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \\
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \\
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,
\]

are all examples of real analytic functions with radius of convergence \( R = \infty \). On the other hand,

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,
\]

is real analytic with radius of convergence \( R = 1 \).

Remark.

Recall that, given a power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \), the radius of convergence is given by \( R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} \).

Definition of ordinary and singular points.

Consider the homogeneous second-order differential equation

\[
y'' + p(x)y' + q(x)y = 0. \tag{3.2}
\]

We say that \( x_0 \) is an ordinary point of the differential equation if \( p(x) \) and \( q(x) \) are (real) analytic at \( x = x_0 \).

If \( p(x) \) or \( q(x) \) is not analytic at \( x = x_0 \), we say that \( x_0 \) is a singular point of the differential equation.

There are two types of singular points. If \( x_0 \) is a singular point such that \((x - x_0)p(x)\) and \((x - x_0)^2q(x)\) are analytic at \( x = x_0 \), we say \( x_0 \) is a regular singular point. Otherwise, \( x_0 \) is an irregular singular point.

In the following sections, we will study the existence of series solutions of (3.2) for ordinary and regular singular points.
3.3 Existence of series solution

Theorem 1. Consider the homogeneous second-order differential equation

\[ y'' + p(x)y' + q(x)y = 0. \]

(a) If \( x_0 \) is an ordinary point for the differential equation, then the solution of the differential equation is represented in a power series, i.e.,

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

for \( |x - x_0| < R \), where \( R \) is the minimum of the radii of convergence of \( p(x) \) and \( q(x) \).

(b) If \( x_0 \) is a regular singular point of the differential equation, then the differential equation has at least one solution of the form

\[ y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n. \]

Here, \( r \) may be complex.

We will discuss part (b) of the theorem in more details in the following section. For now, let us see some simple applications.

Example 1.

Consider the differential equation

\[ y'' - 2xy' + 2\lambda y = 0. \]

The point \( x = 0 \) is an ordinary point of the differential equation. By Theorem 1, we know that we have a power series solution (with radius of convergence \( R = \infty \)). We make the Ansatz

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]
It follows that
\[ y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^{n-1}, \]
\[ y''(x) = \sum_{n=0}^{\infty} n(n - 1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2}. \]

Substituting back in the differential equation gives
\[ \sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} - 2 \sum_{n=1}^{\infty} na_n x^n + 2\lambda \sum_{n=0}^{\infty} a_n x^n = 0. \]

Since the RHS is zero, all the coefficients in the LHS are zero, and hence
\[ a_{n+2} = \frac{2(n - \lambda)}{(n + 2)(n + 1)}a_n, \quad n \geq 0. \]

This is called the recurrence relationship or recursion formula. It follows from the recursion formula that \( a_0 \) determines \( a_{2k}, k \geq 1 \), and \( a_1 \) determines \( a_{2k+1}, k \geq 1 \).\(^2\) When \( a_0 \neq 0 \) and \( a_1 = 0 \), we have the solution
\[ y_1(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}. \]

On the other hand, if \( a_0 = 0 \) and \( a_1 \neq 0 \), we have
\[ y_2(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}. \]

\(^2\)By iterating the recursion formula
\[ a_{2k} = 2^k \frac{(2k - 2 - \lambda)(2k - 4 - \lambda) \cdots \lambda}{(2k)!} a_0 \]
and
\[ a_{2k+1} = 2^k \frac{(2k - 1 - \lambda)(2k - 3 - \lambda) \cdots (1 - \lambda)}{(2k + 1)!} a_1. \]
The solutions $y_1$ and $y_2$ form a fundamental set of solutions, and the most general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Note that if $\lambda$ is an even integer, $y_1$ reduces to a polynomial of order $\lambda$ (i.e., series terminates), while if $\lambda$ is an odd integer, $y_2$ reduces to a polynomial of order $\lambda$.

**Example 2. Legendre equation.**

Consider the differential equation

$$(1 - x^2)y'' - 2xy' + \mu(\mu + 1)y = 0,$$

the so called Legendre equation. This equation appears in problems with spherical symmetry. Dividing by $(1 - x^2)$, we have

$$y'' - \frac{2x}{1 - x^2}y' + \frac{\mu(\mu + 1)}{1 - x^2}y = 0.$$ 

Since $p(x) = -\frac{2x}{1 - x^2}$ and $q(x) = \frac{\mu(\mu + 1)}{1 - x^2}$ are real analytic at $x = 0$ with radius of convergence $R = 1$, we have a series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$ 

Substituting back in the differential equation gives

$$(1 - x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} na_n x^{n-1} + \mu(\mu + 1) \sum_{n=0}^{\infty} a_n x^n = 0.$$ 

It follows

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \mu(\mu + 1)a_n x^n = 0.$$ 

The coefficient of $x^0$ is

$$2 \cdot 1 \cdot a_2 + \mu(\mu + 1)a_0 = 0,$$
which implies

\[ a_2 = -\frac{\mu(\mu + 1)}{2} a_0. \]

The coefficient of \( x^1 \) is

\[ 3 \cdot 2 \cdot a_3 - 2a_1 + \mu(\mu + 1)a_1 = 0, \]

which implies

\[ a_3 = -\frac{(\mu - 1)(\mu + 2)}{3 \cdot 2} a_1. \]

The coefficient of \( x^n, n \geq 2, \)

\[ a_{n+2} = -\frac{(\mu - n)(\mu + n + 1)}{(n + 2)(n + 1)}a_n, \quad n = 0, 1, \ldots \]

Using this recursion formula, \( a_0 \) generates all \( a_{2k}, k \geq 1, \) and \( a_1 \) generates all \( a_{2k+1}, \; k \geq 1. \) The general solution is

\[ y(x) = a_0y_1(x) + a_1y_2(x), \]

where the fundamental solutions are

\[ y_1(x) = 1 - \frac{\mu(\mu + 1)}{2} x^2 + \frac{(\mu - 2)\mu(\mu + 1)(\mu + 3)}{4!} x^4 - \cdots, \]
\[ y_2(x) = x - \frac{(\mu - 1)(\mu + 2)}{3!} x^3 + \frac{(\mu - 3)(\mu - 1)(\mu + 2)(\mu + 4)}{5!} x^5 - \cdots. \]

Both series are convergent for \( |x| < 1. \)

If \( \mu \) is an even integer, \( y_1 \) reduces to a polynomial of degree \( \mu. \) On the other hand, if \( \mu \) is an odd integer, \( y_2 \) reduces to a polynomial of degree \( \mu. \) These polynomials are called Legendre polynomials (an example of orthogonal polynomials, something we will discuss in detail in the future). Suppose \( \mu = m. \) If we choose the normalization

\[ a_m = \frac{(2m)!}{2^m(m!)^2}, \]

we have

\[ a_{m-2k} = \frac{(-1)^k(2m - 2k)!}{2^m k!(m - k)!(m - 2k)!}. \]
The Legendre polynomial is given by

\[ P_m(x) = \sum_{k=0}^{K} \frac{(-1)^k(2m - 2k)!}{2^mk!(m-k)!(m-2k)!} x^{m-2k}, \]

where

\[ K = \begin{cases} \frac{m}{2}, & m \text{ even}, \\ \frac{m-1}{2}, & m \text{ odd}. \end{cases} \]

For example,

\[ P_0(x) = 1, \]
\[ P_1(x) = x, \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1), \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x), \]
\[ \ldots \]

One can show that

\[ P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} ((x^2 - 1)^m). \]

**Example 3.**

Consider the differential equation

\[ x^2 y'' + xy' - y = 0. \]

Dividing by \( x^2 \) gives

\[ y'' + p(x)y' + q(x)y = 0, \]

where \( p(x) = \frac{1}{x} \) and \( q(x) = -\frac{1}{x^2} \). The point \( x = 0 \) is a regular singular point, since \( xp(x) = 1 \) and \( x^2q(x) = -1 \) are analytic at \( x = 0 \). By Theorem 1 (b), there exists at least one series solution of the form

\[ y(x) = x^r \sum_{n=0}^{\infty} a_n x^n. \]
Differentiating with respect to $x$, we have
\[y'(x) = \sum_{n=0}^{\infty} (n + r)a_n x^{n+r-1},\]
\[y''(x) = \sum_{n=0}^{\infty} (n + r)(n + r - 1)x^{n+r-2}.\]

Substituting back in the differential equation gives
\[\sum_{n=0}^{\infty} [(n + r)(n + r - 1) + (n + r) - 1]a_n x^{n+r} = 0,\]
which implies
\[[(n + r)^2 - 1]a_n = (n + r + 1)(n + r - 1)a_n = 0.\]
To get a nontrivial solution, assume $a_0 \neq 0$, and $a_n = 0, n \geq 1$, which implies $r = -1$ or $r = 1$. This gives two fundamental solutions
\[y_1(x) = \frac{1}{x},\]
\[y_2(x) = x.\]
Note that choosing $a_i \neq 0$ and $a_j = 0, j \neq i$, gives the same fundamental solutions, since $r = -1 - i$ or $r = 1 - i$, and $y = a_i x^{r+i}$.

### 3.4 Frobenius method

In this section, we take a closer look at finding solutions in the case of regular singular points of ODEs.

Consider the differential equation
\[y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2} y = 0,\]
where $b(x)$ and $c(x)$ are analytic at $x = 0$,
\[b(x) = \sum_{n=0}^{\infty} b_n x^n,\]
\[c(x) = \sum_{n=0}^{\infty} c_n x^n.\]
We know that the ODE has at least one solution of the form

\[ y(x) = x^r \sum_{n=0}^{\infty} a_n x^n. \]

Substituting back in the differential equation, the coefficient of \( x^0 \) is

\[ r(r - 1) + b_0 r + c_0 = 0. \]

This quadratic equation in \( r \) is called the *indicial equation*, and its solution gives the value of \( r \) (which may be complex). We distinguish three cases depending on the roots.

(i) *Distinct roots \( r_1 \) and \( r_2 \) that do not differ by an integer.* In this case, the fundamental solutions are

\[ y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \]
\[ y_2(x) = x^{r_2} \sum_{n=0}^{\infty} A_n x^n. \]

(ii) *Double roots \( r_1 = r_2 = r.\) In this case, the fundamental solutions are

\[ y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n, \]
\[ y_2(x) = y_1(x) \ln x + x^r \sum_{n=1}^{\infty} A_n x^n, \quad x > 0. \]

(iii) *Distinct roots \( r_1 \) and \( r_2 \) that differ by an integer.* Without loss of generality, assume \( r_1 > r_2.\) In this case, the fundamental solutions are

\[ y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \]
\[ y_2(x) = ky_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} A_n x^n, \quad x > 0. \]

for some constant \( k \) that might be zero.
3.5 Bessel’s equation

We now apply the Frobenius method to find solutions of the Bessel equation, which appears in studying problems with cylindrical symmetry.

Bessel’s equation is given by

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0.$$ 

Since the point $x = 0$ is a regular singular point of the differential equation, it follows from Theorem 1 that there exists at least one solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n.$$

Substituting back in the differential equation, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

The indicial equation corresponds to setting the coefficient of $x^0$ to zero, which is

$$r^2 - \nu^2 = (r - \nu)(r + \nu) = 0.$$ 

Without loss of generality, let $\nu > 0$.

We consider first the case $r = \nu$. The coefficient of $x^1$ is

$$(2\nu + 1)a_1 = 0,$$

which implies $a_1 = 0$. For $n \geq 2$, the coefficient of $x^n$ is

$$(n + 2\nu)a_n + a_{n-2} = 0.$$ 

It follows from this recursion formula and $a_1 = 0$ that $a_{2k+1} = 0$, $k \geq 1$. Furthermore,

$$a_{2k} = -\frac{1}{2^k k + \nu} a_{2k-2}, \quad k \geq 1.$$ 

Iterating this relationship, we have

$$a_{2k} = \frac{(-1)^k}{2^k k! (\nu + 1)(\nu + 2) \cdots (\nu + k)} a_0, \quad k \geq 1.$$ 

We will discuss different cases depending on whether $\nu$ is an integer or not.
3.5.1 Bessel function of first kind of order $n$

If $\nu = n$ is an integer,

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(n+1)(n+2)\cdots(n+k)}a_0, \quad k \geq 1.$$  

Choosing the normalization

$$a_0 = \frac{1}{2^n n!}$$

gives

$$a_{2k} = \frac{(-1)^k}{2^{2k+n}k!(n+k)!}, \quad k \geq 1.$$  

The resulting function is called Bessel function of first kind of order $n$,

$$J_n(x) = x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+n}k!(n+k)!}.$$  

We note that the series resembles a cosine. Indeed, as $x \to \infty$,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{n\pi}{2} - \frac{\pi}{4}).$$  

3.5.2 Bessel function of first kind

We first generalize the notion of factorial to the case of positive real numbers. Let $f(n) = n!$. Then the key property satisfied by $f$ is $f(n+1) = (n+1)f(n)$. We define the $\Gamma$-function as

$$\Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1}dt, \quad \nu > 0.$$  

Integrating by parts, we have

$$\Gamma(\nu + 1) = \nu \Gamma(\nu).$$  

Furthermore,

$$\Gamma(1) = 1$$  

and

$$\Gamma(n+1) = n!.$$
Figure 3.1: A plot of $J_0(x)$.

Figure 3.2: A plot of $J_1(x)$.
This is exactly the function we are after.

Choosing

\[ a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)} \]

yields

\[ a_{2k} = \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu + k + 1)}, \quad k \geq 1. \]

The resulting function is called Bessel function of first kind

\[ J_\nu(x) = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\nu} k! \Gamma(k + \nu + 1)}. \]

Note that for \( \nu \) different than an integer, one can generate a new solution by replacing \( \nu \) with \( -\nu \).\(^3\) The second fundamental solution is given by

\[ J_{-\nu}(x) = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k-\nu} k! \Gamma(k - \nu + 1)}, \]

and a general solution is given by

\[ c_1 J_\nu(x) + c_2 J_{-\nu}(x). \]

However, this argument fails when \( \nu \) is an integer, as we shall see below.

### 3.5.3 Bessel function of second kind

When \( \nu = -n \) is a negative integer,

\[ J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n} k! (k - n)!} \]

\[ = \sum_{m=0}^{\infty} \frac{(-1)^{n+m} x^{2m+n}}{2^{2m+n} (n + m)! m!} \]

\[ = (-1)^n J_n(x). \]

\(^3\)To find \( \Gamma(\alpha) \) for negative noninteger \( \alpha \), use the recursion formula for the \( \Gamma \)-function to obtain the relationship

\[ \Gamma(\alpha) = \frac{\Gamma(k + \alpha + 1)}{\alpha(\alpha + 1) \cdots (\alpha + k)}, \]

where \( k \) is the smallest integer such that \( k + \alpha + 1 > 0 \).
Therefore, \( J_n \) and \( J_{-n} \) are linearly dependent, and we need to work harder to get a second fundamental solution.

Consider first the case \( n = 0 \), which corresponds to the case of a double root. Bessel’s equation becomes

\[
xy'' + y' + xy = 0.
\]

Using Frobenius method, we make the Ansatz

\[
y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m.
\]

Substituting back in the differential equation and using the fact that \( J_0 \) satisfies the differential equation gives

\[
2J_0'(x) + \sum_{m=0}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.
\]

Since \( J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!} \), we have

\[
\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2}m!(m-1)!} + \sum_{m=0}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.
\]

The coefficient of \( x^0 \) is

\[
A_1 = 0.
\]

We discuss terms with odd and even powers independently. The coefficient of \( x^{2k}, \ k \geq 1 \), is

\[
(2k + 1)^2 A_{2k+1} + A_{2k-1} = 0.
\]

Since \( A_1 = 0 \), it follows that \( A_{2k+1} = 0, \ k \geq 1 \). We consider now the case of odd powers. The coefficient of \( x^1 \) is

\[
-1 + 4A_2 = 0,
\]

which implies \( A_2 = \frac{1}{4} \). The coefficient of \( x^{2k+1} \) is

\[
\frac{(-1)^{k+1}}{2^{2k}(k+1)!k!} + (2k + 2)^2 A_{2k+2} + A_{2k} = 0.
\]
This gives

$$A_{2k} = \frac{(-1)^{k-1}}{2^{2k} (k!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right).$$

It follows that

$$y_2(x) = J_0(x) \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}h_k}{2^{2k} (k!)^2} x^{2k},$$

where $h_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. Recall that we can take any linear combination to generate a solution. Let $\gamma = \lim_{s \to \infty} (h_s - \ln s)$, the Euler constant. The Bessel function of second kind of order 0 is

$$Y_0 = \frac{2}{\pi} [J_0(x)(\ln \frac{x}{2} + \gamma) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}h_k}{2^{2k} (k!)^2} x^{2k}].$$

Note that $Y_0(x)$ behaves like $\ln x$ for small $x$.

For the case $\nu = -n$, we can similarly apply the Frobenius method to find the second fundamental solution. Generally, the Bessel function of second kind is given by

$$Y_\nu(x) = \frac{1}{\sin(\nu \pi)} (J_\nu(x) \cos(\nu \pi) - J_{-\nu}(x)).$$
and

\[ Y_n(x) = \lim_{\nu \to n} Y_\nu(x). \]

### 3.5.4 Properties of the Bessel function

In this subsection, we list the properties of the Bessel function of first kind, which follow directly from the series expansion (see homework assignment 3).

(a) \((x^\nu J_\nu(x))^' = x^\nu J_{\nu-1}(x)\).

(b) \((x^{-\nu} J_\nu(x))^' = -x^{-\nu} J_{\nu+1}(x)\).

(c) \(J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)\).

(d) \(J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)\).