Chapter 9

Filter Design

9.1 INTRODUCTION

This chapter considers the problem of designing a digital filter. The design process begins with the filter specifications, which may include constraints on the magnitude and/or phase of the frequency response, constraints on the unit sample response or step response of the filter, specification of the type of filter (e.g., FIR or IIR), and the filter order. Once the specifications have been defined, the next step is to find a set of filter coefficients that produce an acceptable filter. After the filter has been designed, the last step is to implement the system in hardware or software, quantizing the filter coefficients if necessary, and choosing an appropriate filter structure (Chap. 8).

9.2 FILTER SPECIFICATIONS

Before a filter can be designed, a set of filter specifications must be defined. For example, suppose that we would like to design a low-pass filter with a cutoff frequency \( \omega_c \). The frequency response of an ideal low-pass filter with linear phase and a cutoff frequency \( \omega_c \) is

\[
H_d(e^{j\omega}) = \begin{cases} 
  e^{-j\alpha \omega} & |\omega| \leq \omega_c \\
  0 & \omega_c < |\omega| \leq \pi
\end{cases}
\]

which has a unit sample response

\[
h_d(n) = \frac{\sin(n - \alpha)\omega_c}{\pi(n - \alpha)}
\]

Because this filter is unrealizable (noncausal and unstable), it is necessary to relax the ideal constraints on the frequency response and allow some deviation from the ideal response. The specifications for a low-pass filter will typically have the form

\[
1 - \delta_p < |H(e^{j\omega})| \leq 1 + \delta_p \quad 0 \leq |\omega| < \omega_p \\
|H(e^{j\omega})| \leq \delta_s \quad \omega_s \leq |\omega| < \pi
\]

as illustrated in Fig. 9-1. Thus, the specifications include the passband cutoff frequency, \( \omega_p \), the stopband cutoff frequency, \( \omega_s \), the passband deviation, \( \delta_p \), and the stopband deviation, \( \delta_s \). The passband and stopband deviations
are often given in decibels (dB) as follows:

\[ \alpha_p = -20 \log(1 - \delta_p) \]

and

\[ \alpha_s = -20 \log(\delta_s) \]

The interval \([\omega_p, \omega_s]\) is called the transition band.

Once the filter specifications have been defined, the next step is to design a filter that meets these specifica-
tions.

9.3 **FIR FILTER DESIGN**

The frequency response of an \(N\)th-order causal FIR filter is

\[ H(e^{j\omega}) = \sum_{n=0}^{N} h(n)e^{-jn\omega} \]

and the design of an FIR filter involves finding the coefficients \(h(n)\) that result in a frequency response that satisfies a given set of filter specifications. FIR filters have two important advantages over IIR filters. First, they are guaranteed to be stable, even after the filter coefficients have been quantized. Second, they may be easily constrained to have (generalized) linear phase. Because FIR filters are generally designed to have linear phase, in the following we consider the design of linear phase FIR filters.

9.3.1 **Linear Phase FIR Design Using Windows**

Let \(h_d(n)\) be the unit sample response of an ideal frequency selective filter with linear phase,

\[ H_d(e^{j\omega}) = A(e^{j\omega})e^{-j\omega_0 - \beta} \]

Because \(h_d(n)\) will generally be infinite in length, it is necessary to find an FIR approximation to \(H_d(e^{j\omega})\). With the window design method, the filter is designed by windowing the unit sample response,

\[ h(n) = h_d(n)w(n) \]

where \(w(n)\) is a finite-length window that is equal to zero outside the interval \(0 \leq n \leq N\) and is symmetric about its midpoint:

\[ w(n) = w(N - n) \]

The effect of the window on the frequency response may be seen from the complex convolution theorem,

\[ H(e^{j\omega}) = \frac{1}{2\pi} H_d(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta \]

Thus, the ideal frequency response is smoothed by the discrete-time Fourier transform of the window, \(W(e^{j\omega})\).

There are many different types of windows that may be used in the window design method, a few of which are listed in Table 9-1.

How well the frequency response of a filter designed with the window design method approximates a desired response, \(H_d(e^{j\omega})\), is determined by two factors (see Fig. 9-2):

1. The width of the main lobe of \(W(e^{j\omega})\).
2. The peak side-lobe amplitude of \(W(e^{j\omega})\).
Ideally, the main-lobe width should be narrow, and the side-lobe amplitude should be small. However, for a fixed-length window, these cannot be minimized independently. Some general properties of windows are as follows:

1. As the length $N$ of the window increases, the width of the main lobe decreases, which results in a decrease in the transition width between passbands and stopbands. This relationship is given approximately by

$$N \Delta f = c \quad (9.1)$$

where $\Delta f$ is the transition width, and $c$ is a parameter that depends on the window.

2. The peak side-lobe amplitude of the window is determined by the shape of the window, and it is essentially independent of the window length.

3. If the window shape is changed to decrease the side-lobe amplitude, the width of the main lobe will generally increase.

Listed in Table 9.2 are the side-lobe amplitudes of several windows along with the approximate transition width and stopband attenuation that results when the given window is used to design an $N$th-order low-pass filter.

### Table 9-1  Some Common Windows

<table>
<thead>
<tr>
<th>Window</th>
<th>$w(n)$</th>
</tr>
</thead>
</table>
| Rectangular | \[
|           | \begin{cases}
|           | 1 & 0 \leq n \leq N \\
|           | 0 & \text{else} \\
|          | \end{cases}                                                             |
| Hanning\(^1\) | \[
|           | \begin{cases}
|           | 0.5 - 0.5 \cos \left( \frac{2\pi n}{N} \right) & 0 \leq n \leq N \\
|           | 0 & \text{else} \\
|          | \end{cases}                                                             |
| Hamming  | \[
|           | \begin{cases}
|           | 0.54 - 0.46 \cos \left( \frac{2\pi n}{N} \right) & 0 \leq n \leq N \\
|           | 0 & \text{else} \\
|          | \end{cases}                                                             |
| Blackman | \[
|           | \begin{cases}
|           | 0.42 - 0.5 \cos \left( \frac{2\pi n}{N} \right) + 0.08 \cos \left( \frac{4\pi n}{N} \right) & 0 \leq n \leq N \\
|           | 0 & \text{else} \\
|          | \end{cases}                                                             |

\(^1\)In the literature, this window is also called a Hann window or a von Hann window.
**EXAMPLE 9.3.1** Suppose that we would like to design an FIR linear phase low-pass filter according to the following specifications:

Table 9-2 The Peak Side-Lobe Amplitude of Some Common Windows and the Approximate Transition Width and Stopband Attenuation of an Nth-Order Low-Pass Filter Designed Using the Given Window.

<table>
<thead>
<tr>
<th>Window</th>
<th>Side-Lobe Amplitude (dB)</th>
<th>Transition Width (Δf)</th>
<th>Stopband Attenuation (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>−13</td>
<td>0.9/N</td>
<td>−21</td>
</tr>
<tr>
<td>Hanning</td>
<td>−31</td>
<td>3.1/N</td>
<td>−44</td>
</tr>
<tr>
<td>Hamming</td>
<td>−41</td>
<td>3.3/N</td>
<td>−53</td>
</tr>
<tr>
<td>Blackman</td>
<td>−57</td>
<td>5.5/N</td>
<td>−74</td>
</tr>
</tbody>
</table>

For a stopband attenuation of 20 log(0.01) = −40 dB, we may use a Hanning window. Although we could also use a Hamming or a Blackman window, these windows would overdesign the filter and produce a larger stopband attenuation at the expense of an increase in the transition width. Because the specification calls for a transition width of Δω = ωt − ωp = 0.02π, or Δf = 0.01, with

\[
N Δf = 3.1
\]

for a Hanning window (see Table 9.2), an estimate of the required filter order is

\[
N = \frac{3.1}{Δf} = 310
\]

The last step is to find the unit sample response of the ideal low-pass filter that is to be windowed. With a cutoff frequency of \(ωc = (ωt + ωp)/2 = 0.2π\), and a delay of \(d = N/2 = 155\), the unit sample response is

\[
h_d(n) = \frac{\sin[0.2π(n − 155)]}{(n − 155)π}
\]

In addition to the windows listed in Table 9-1, Kaiser developed a family of windows that are defined by

\[
w(n) = \frac{I_0[\beta(1 − [(n − α)/α^2])^{1/2}]}{I_0(\beta)} \quad 0 \leq n \leq N
\]

where \(α = N/2\), and \(I_0(\cdot)\) is a zeroth-order modified Bessel function of the first kind, which may be easily generated using the power series expansion

\[
I_0(x) = 1 + \sum_{k=1}^{∞} \left[\frac{(x/2)^k}{k!}\right]^2
\]

The parameter \(β\) determines the shape of the window and thus controls the trade-off between main-lobe width and side-lobe amplitude. A Kaiser window is nearly optimum in the sense of having the most energy in its main lobe for a given side-lobe amplitude. Table 9-3 illustrates the effect of changing the parameter \(β\).

There are two empirically derived relationships for the Kaiser window that facilitate the use of these windows to design FIR filters. The first relates the stopband ripple of a low-pass filter, \(α_s = −20 \log(δ_s)\), to the parameter \(β\),

\[
β = \begin{cases} 
0.1102(α_s − 8.7) & α_s > 50 \\
0.5842(α_s − 21)^{0.4} + 0.07886(α_s − 21) & 21 \leq α_s \leq 50 \\
0.0 & α_s < 21
\end{cases}
\]
The second relates $N$ to the transition width $\Delta f$ and the stopband attenuation $\alpha_s$,

$$N = \frac{\alpha_s - 7.95}{14.36 \Delta f} \quad \alpha_s \geq 21 \quad (9.2)$$

Note that if $\alpha_s < 21$ dB, a rectangular window may be used ($\beta = 0$), and $N = 0.9/\Delta f$.

**EXAMPLE 9.3.2** Suppose that we would like to design a low-pass filter with a cutoff frequency $\omega_c = \pi/4$, a transition width $\Delta \omega = 0.02\pi$, and a stopband ripple $\delta_s = 0.01$. Because $\alpha_s = -20 \log(0.01) = -40$, the Kaiser window parameter is

$$\beta = 0.5842(40 - 21)^{0.4} + 0.07886(40 - 21) = 3.4$$

With $\Delta f = \Delta \omega/2\pi = 0.01$, we have

$$N = \frac{40 - 7.95}{14.36 \cdot (0.01)} = 224$$

Therefore,

$$h(n) = h_d(n)w(n)$$

where

$$h_d(n) = \frac{\sin[(n - 112)\pi/4]}{(n - 112)\pi}$$

is the unit sample response of the ideal low-pass filter.

Although it is simple to design a filter using the window design method, there are some limitations with this method. First, it is necessary to find a closed-form expression for $h_d(n)$ (or it must be approximated using a very long DFT). Second, for a frequency selective filter, the transition widths between frequency bands, and the ripples within these bands, will be approximately the same. As a result, the window design method requires that the filter be designed to the tightest tolerances in all of the bands by selecting the smallest transition width and the smallest ripple. Finally, window design filters are not, in general, optimum in the sense that they do not have the smallest possible ripple for a given filter order and a given set of cutoff frequencies.

### Frequency Sampling Filter Design

Another method for FIR filter design is the frequency sampling approach. In this approach, the desired frequency response, $H_d(e^{j\omega})$, is first uniformly sampled at $N$ equally spaced points between 0 and $2\pi$:

$$H(k) = H_d(e^{j2\pi k/N}) \quad k = 0, 1, \ldots, N - 1$$
These frequency samples constitute an $N$-point DFT, whose inverse is an FIR filter of order $N - 1$:

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j2\pi nk/N} \quad 0 \leq n \leq N - 1$$

The relationship between $h(n)$ and $h_d(n)$ (see Chap. 3) is

$$h(n) = \sum_{k=-\infty}^{\infty} h_d(n + kN) \quad 0 \leq n \leq N - 1$$

Although the frequency samples match the ideal frequency response exactly, there is no control on how the samples are interpolated between the samples. Because filters designed with the frequency sampling method are not generally very good, this method is often modified by introducing one or more transition samples as illustrated in Fig. 9-3. These transition samples are optimized in an iterative manner to maximize the stopband attenuation or minimize the passband ripple.

![Fig. 9-3. Introducing a transition sample with an amplitude of $A_1$ in the frequency sampling method.](image)

### 9.3.3 Equiripple Linear Phase Filters

The design of an FIR low-pass filter using the window design technique is simple and generally results in a filter with relatively good performance. However, in two respects, these filters are not optimal:

1. First, the passband and stopband deviations, $\delta_p$ and $\delta_s$, are approximately equal. Although it is common to require $\delta_s$ to be much smaller than $\delta_p$, these parameters cannot be independently controlled in the window design method. Therefore, with the window design method, it is necessary to overdesign the filter in the passband in order to satisfy the stricter requirements in the stopband.

2. Second, for most windows, the ripple is not uniform in either the passband or the stopband and generally decreases when moving away from the transition band. Allowing the ripple to be uniformly distributed over the entire band would produce a smaller peak ripple.

An equiripple linear phase filter, on the other hand, is optimal in the sense that the magnitude of the ripple is minimized in all bands of interest for a given filter order, $N$. In the following discussion, we consider the design of a type I linear phase filter. The results may be easily modified to design other types of linear phase filters.

The frequency response of an FIR linear phase filter may be written as

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega}$$  \hspace{1cm} (9.3)
where the amplitude, \( A(e^{j\omega}) \), is a real-valued function of \( \omega \). For a Type I linear phase filter,

\[
h(n) = h(N - n)
\]

where \( N \) is an even integer. The symmetry of \( h(n) \) allows the frequency response to be expressed as

\[
A(e^{j\omega}) = \sum_{k=0}^{L} a(k) \cos(k\omega)
\]

(9.4)

where \( L = N/2 \) and

\[
a(0) = h\left(\frac{N}{2}\right)
\]

\[
a(k) = h\left(k + \frac{N}{2}\right) \quad k = 1, 2, \ldots, \frac{N}{2}
\]

The terms \( \cos(k\omega) \) may be expressed as a sum of powers of \( \cos \omega \) in the form

\[
\cos(k\omega) = T_k(\cos \omega)
\]

where \( T_k(x) \) is a \( k \)th-order Chebyshev polynomial [see Eq. (9.9)]. Therefore, Eq. (9.4) may be written as

\[
A(e^{j\omega}) = \sum_{k=0}^{L} a(k)(\cos \omega)^k
\]

Thus, \( A(e^{j\omega}) \) is an \( L \)th-order polynomial in \( \cos \omega \).

With \( A_d(e^{j\omega}) \) a desired amplitude, and \( W(e^{j\omega}) \) a positive weighting function, let

\[
E(e^{j\omega}) = W(e^{j\omega})[A_d(e^{j\omega}) - A(e^{j\omega})]
\]

be a weighted approximation error. The equiripple filter design problem thus involves finding the coefficients \( a(k) \) that minimize the maximum absolute value of \( E(e^{j\omega}) \) over a set of frequencies, \( \mathcal{F} \),

\[
\min_{a(k)} \left\{ \max_{\omega \in \mathcal{F}} |E(e^{j\omega})| \right\}
\]

For example, to design a low-pass filter, the set \( \mathcal{F} \) will be the frequencies in the passband, \([0, \omega_p]\), and the stopband, \([\omega_s, \pi]\), as illustrated in Fig. 9-4. The transition band, \((\omega_p, \omega_s)\), is a don’t care region, and it is not

![Fig. 9-4. The set \( R \) in the equiripple filter design problem, consisting of the passband \([0, \omega_p]\) and the stopband \([\omega_p, \omega_s]\). The transition band \((\omega_p, \omega_s)\) is a don’t care region.](image)

considered in the minimization of the weighted error. The solution to this optimization problem is given in the alternation theorem, which is as follows:

**Alternation Theorem:** Let $\mathcal{F}$ be a union of closed subsets over the interval $[0, \pi]$. For a positive weighting function $W(e^{j\omega})$, a necessary and sufficient condition for

$$A(e^{j\omega}) = \sum_{k=0}^{L} a(k) \cos(k\omega)$$

to be the unique function that minimizes the maximum value of the weighted error $|E(e^{j\omega})|$ over the set $\mathcal{F}$ is that the $E(e^{j\omega})$ have at least $L+2$ *alternations*. That is to say, there must be at least $L+2$ extremal frequencies,

$$\omega_0 < \omega_1 < \cdots < \omega_{L+1}$$

over the set $\mathcal{F}$ such that

$$E(e^{j\omega_k}) = -E(e^{j\omega_{k+1}}) \quad k = 0, 1, \ldots, L$$

and

$$|E(e^{j\omega_k})| = \max_{\omega \in \mathcal{F}} |E(e^{j\omega})| \quad k = 0, 1, \ldots, L + 1$$

Thus, the alternation theorem states that the optimum filter is equiripple. Although the alternation theorem specifies the minimum number of extremal frequencies (or ripples) that the optimum filter must have, it may have more. For example, a low-pass filter may have either $L+2$ or $L+3$ extremal frequencies. A low-pass filter with $L+3$ extrema is called an *extraripple filter*.

From the alternation theorem, it follows that

$$W(e^{j\omega_k})[A_d(e^{j\omega_k}) - A(e^{j\omega_k})] = (-1)^k \epsilon \quad k = 0, 1, \ldots, L + 1$$

where

$$\epsilon = \pm \max_{\omega \in \mathcal{F}} |E(e^{j\omega})|$$

is the maximum absolute weighted error. These equations may be written in matrix form in terms of the unknowns $a(0), \ldots, a(L)$ and $\epsilon$ as follows:

$$\begin{bmatrix}
1 & \cos(\omega_0) & \cdots & \cos(L\omega_0) \\
1 & \cos(\omega_1) & \cdots & \cos(L\omega_1) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \cos(\omega_L) & \cdots & \cos(L\omega_L) \\
1 & \cos(\omega_{L+1}) & \cdots & \cos(L\omega_{L+1})
\end{bmatrix}
\begin{bmatrix}
1/W(e^{j\omega_0}) \\
-1/W(e^{j\omega_1}) \\
\vdots \\
(-1)^L/W(e^{j\omega_L}) \\
(-1)^{L+1}/W(e^{j\omega_{L+1}})
\end{bmatrix}
\begin{bmatrix}
a(0) \\
a(1) \\
\vdots \\
a(L) \\
\epsilon
\end{bmatrix} =
\begin{bmatrix}
A_d(e^{j\omega_0}) \\
A_d(e^{j\omega_1}) \\
\vdots \\
A_d(e^{j\omega_L}) \\
A_d(e^{j\omega_{L+1}})
\end{bmatrix} \quad (9.5)$$

Given the extremal frequencies, these equations may be solved for $a(0), \ldots, a(L)$ and $\epsilon$. To find the extremal frequencies, there is an efficient iterative procedure known as the Parks-McClellan algorithm, which involves the following steps:

1. Guess an initial set of extremal frequencies.
2. Find $\epsilon$ by solving Eq. (9.5). The value of $\epsilon$ has been shown to be

$$\epsilon = \frac{\sum_{k=0}^{L+1} b(k) D(e^{j\omega_k})}{\sum_{k=0}^{L+1} (-1)^k b(k)/W(e^{j\omega_k})}$$
where
\[ b(k) = \prod_{i=1,i\neq k}^{L+1} \frac{1}{\cos(\omega_k) - \cos(\omega_i)} \]

3. Evaluate the weighted error function over the set \( F \) by interpolating between the extremal frequencies using the Lagrange interpolation formula.

4. Select a new set of extremal frequencies by choosing the \( L + 2 \) frequencies for which the interpolated error function is maximum.

5. If the extremal frequencies have changed, repeat the iteration from step 2.

A design formula that may be used to estimate the equiripple filter order for a low-pass filter with a transition width \( \Delta f \), passband ripple \( \delta_p \), and stopband ripple \( \delta_s \) is

\[ N = \frac{-10 \log(\delta_p \delta_s) - 13}{14.6 \Delta f} \quad (9.6) \]

**EXAMPLE 9.3.3** Suppose that we would like to design an equiripple low-pass filter with a passband cutoff frequency \( \omega_p = 0.3\pi \), a stopband cutoff frequency \( \omega_s = 0.35\pi \), a passband ripple of \( \delta_p = 0.01 \), and a stopband ripple of \( \delta_s = 0.001 \). Estimating the filter using Eq. (9.6), we find

\[ N = \frac{-10 \log(\delta_p \delta_s) - 13}{14.6 \Delta f} = 102 \]

Because we want the ripple in the stopband to be 10 times smaller than the ripple in the passband, the error must be weighted using the weighting function

\[ W(e^{j\omega}) = \begin{cases} 
1 & 0 \leq |\omega| \leq 0.3\pi \\
10 & 0.35\pi \leq |\omega| \leq \pi 
\end{cases} \]

Using the Parks-McClellan algorithm to design the filter, we obtain a filter with the frequency response magnitude shown below.

### 9.4 IIR FILTER DESIGN

There are two general approaches used to design IIR digital filters. The most common is to design an analog IIR filter and then map it into an equivalent digital filter because the art of analog filter design is highly advanced. Therefore, it is prudent to consider optimal ways for mapping these filters into the discrete-time domain. Furthermore, because there are powerful design procedures that facilitate the design of analog filters, this approach
to IIR filter design is relatively simple. The second approach to design IIR digital filters is to use an algorithmic design procedure, which generally requires the use of a computer to solve a set of linear or nonlinear equations. These methods may be used to design digital filters with arbitrary frequency response characteristics for which no analog filter prototype exists or to design filters when other types of constraints are imposed on the design.

In this section, we consider the approach of mapping analog filters into digital filters. Initially, the focus will be on the design of digital low-pass filters from analog low-pass filters. Techniques for transforming these designs into more general frequency selective filters will then be discussed.

9.4.1 Analog Low-Pass Filter Prototypes

To design an IIR digital low-pass filter from an analog low-pass filter, we must first know how to design an analog low-pass filter. Historically, most analog filter approximation methods were developed for the design of passive systems having a gain less than or equal to 1. Therefore, a typical set of specifications for these filters is as shown in Fig. 9-5(a), with the passband specifications having the form

\[ 1 - \delta_p \leq |H_a(j\Omega)| \leq 1 \]

Another convention that is commonly used is to describe the passband and stopband constraints in terms of the parameters \( \epsilon \) and \( A \) as illustrated in Fig. 9-5(b). Two auxiliary parameters of interest are the discrimination factor,

\[ d = \left( \frac{(1 - \delta_p)^{-2} - 1}{\delta_c^{-2} - 1} \right)^{1/2} = \frac{\epsilon}{\sqrt{A^2 - 1}} \]

and the selectivity factor

\[ k = \frac{\Omega_p}{\Omega_s} \]

The three most commonly used analog low-pass filters are the Butterworth, Chebyshev, and elliptic filters. These filters are described below.

Butterworth Filter

A low-pass Butterworth filter is an all-pole filter with a squared magnitude response given by

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/j\Omega_c)^{2N}} \]
The parameter $N$ is the order of the filter (number of poles in the system function), and $\Omega_c$ is the 3-dB cutoff frequency. The magnitude of the frequency response may also be written as

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2(j\Omega/j\Omega_p)^{2N}}$$

where

$$\epsilon = \left(\frac{\Omega_p}{\Omega_c}\right)^N$$

The frequency response of the Butterworth filter decreases monotonically with increasing $\Omega$, and as the filter order increases, the transition band becomes narrower. These properties are illustrated in Fig. 9-6, which shows $|H_a(j\Omega)|$ for Butterworth filters of orders $N = 2, 4, 8,$ and $12$. Because

$|H_a(j\Omega)|^2 = H_a(s)H_a(-s)|_{s=j\Omega}$

from the magnitude-squared function, we may write

$$G_a(s) = H_a(s)H_a(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}$$

Therefore, the poles of $G_a(s)$ are located at $2N$ equally spaced points around a circle of radius $\Omega_c$,

$$s_k = (-1)^{1/2N} (j\Omega_c) = \Omega_c \exp\left\{j\frac{(N + 1 + 2k)\pi}{2N}\right\} \quad k = 0, 1, \ldots, 2N - 1 \quad (9.7)$$

and are symmetrically located about the $j\Omega$-axis. Figure 9-7 shows these pole positions for $N = 6$ and $N = 7$. The system function, $H_a(s)$, is then formed from the $N$ roots of $H_a(s)H_a(-s)$ that lie in the left-half $s$-plane. For a normalized Butterworth filter with $\Omega_c = 1$, the system function has the form

$$H_a(s) = \frac{1}{A_N(s)} = \frac{1}{s^N + a_Ns^{N-1} + \cdots + a_1s + a_N} \quad (9.8)$$

Table 9-4 lists the coefficients of $A_N(s)$ for $1 \leq N \leq 8$. Given $\Omega_p$, $\Omega_c$, $\delta_p$, and $\delta_c$, the steps involved in designing a Butterworth filter are as follows:

1. Find the values for the selectivity factor, $k$, and the discrimination factor, $d$, from the filter specifications.
2. Determine the order of the filter required to meet the specifications using the design formula

$$N \geq \frac{\log d}{\log k}$$
(a) Order $N = 6$.  
(b) Order $N = 7$.  

Fig. 9-7.  The poles of $H_a(a)H_a(-s)$ for a Butterworth filter of order $N = 6$ and $N = 7$. 

<table>
<thead>
<tr>
<th>$N$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
<th>$a_7$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
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<td>2.0000</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.6131</td>
<td>3.4142</td>
<td>2.6131</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3.2361</td>
<td>5.2361</td>
<td>5.2361</td>
<td>3.2361</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.8637</td>
<td>7.4641</td>
<td>9.1416</td>
<td>7.4641</td>
<td>3.8637</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Set the 3-dB cutoff frequency, $\Omega_c$, to any value in the range 
   
   $$\Omega_p \left[ (1 - \delta_p)^{-2} - 1 \right]^{-1/2N} \leq \Omega_c \leq \Omega_p \left[ \delta_c^{-2} - 1 \right]^{-1/2N}$$

4. Synthesize the system function of the Butterworth filter from the poles of

   $$G_a(s) = H_a(s)H_a(-s) = \frac{1}{1 + (s/\Omega_c)^{2N}}$$

   that lie in the left-half $s$-plane. Thus,

   $$H_a(s) = \prod_{k=0}^{N-1} \frac{-s_k}{s - s_k}$$

   where 

   $$s_k = \Omega_c \exp \left\{ j \frac{(N + 1 + 2k)\pi}{2N} \right\} \quad k = 0, 1, \ldots, N - 1$$

**EXAMPLE 9.4.1** Let us design a low-pass Butterworth filter to meet the following specifications:

   $$f_p = 6 \text{ kHz} \quad f_s = 10 \text{ kHz} \quad \delta_p = \delta_s = 0.1$$
First, we compute the discrimination and selectivity factors:

\[ d = \left[ \frac{(1 - \delta)^{-2} - 1}{\delta^2 - 1} \right]^{1/2} = 0.0487 \quad k = \frac{\Omega_p}{\Omega_s} = \frac{f_p}{f_s} = 0.6 \]

Because

\[ N \geq \frac{\log d}{\log k} = 5.92 \]

it follows that the minimum filter order is \( N = 6 \). With

\[ f_p(1 - \delta)^{-2} - 1]^{-1/2N} = 6770 \]

and

\[ f_s[\delta^{-2} - 1]^{-1/2N} = 6819 \]

the center frequency, \( f_c \), may be any value in the range

\[ 6770 \leq f_c \leq 6819 \]

The system function of the Butterworth filter may then be found using Eq. (9.8) by first constructing a sixth-order normalized Butterworth filter from Table 9-4,

\[ H_n(s) = \frac{1}{s^6 + 3.8637s^5 + 7.4641s^4 + 9.1416s^3 + 7.4641s^2 + 3.8637s + 1} \]

and then replacing \( s \) with \( s/\Omega_c \) so that the cutoff frequency is \( \Omega_c \) instead of unity (see Sec. 9.4.3).

Chebyshev Filters

Chebyshev filters are defined in terms of the Chebyshev polynomials:

\[ T_N(x) = \begin{cases} \cos(N \cos^{-1} x) & |x| \leq 1 \\ \cosh(N \cosh^{-1} x) & |x| > 1 \end{cases} \quad (9.9) \]

These polynomials may be generated recursively as follows,

\[ T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \quad k \geq 1 \]

with \( T_0(x) = 1 \) and \( T_1(x) = x \). The following properties of the Chebyshev polynomials follow from Eq. (9.9):

1. For \(|x| \leq 1\) the polynomials are bounded by 1 in magnitude, \(|T_N(x)| \leq 1\), and oscillate between \( \pm 1 \). For \(|x| > 1\), the polynomials increase monotonically with \( x \).
2. \( T_N(1) = 1 \) for all \( N \).
3. \( T_N(0) = \pm 1 \) for \( N \) even, and \( T_N(0) = 0 \) for \( N \) odd.
4. All of the roots of \( T_N(x) \) are in the interval \(-1 \leq x \leq 1\).

There are two types of Chebyshev filters. A type I Chebyshev filter is all-pole with an equiripple passband and a monotonically decreasing stopband. The magnitude of the frequency response is

\[ |H_n(j\Omega)|^2 = \frac{1}{1 + \epsilon^2T_N^2(\Omega/\Omega_p)} \]

where \( N \) is the order of the filter, \( \Omega_p \) is the passband cutoff frequency, and \( \epsilon \) is a parameter that controls the passband ripple amplitude. Because \( T_N^2(\Omega/\Omega_p) \) varies between 0 and 1 for \(|\Omega| < \Omega_p\), \( |H_n(j\Omega)|^2 \) oscillates between 1 and \( 1/(1 + \epsilon^2) \). As the order of the filter increases, the number of oscillations (ripples) in the passband increases, and the transition width between the passband and stopband becomes narrower. Examples are given in Fig. 9-8 for \( N = 5, 6 \).
The system function of a type I Chebyshev filter has the form

\[ H_a(s) = H_a(0) \prod_{k=1}^{N-1} \frac{-s_k}{s - s_k} \]

where \( H_a(0) = (1 - \epsilon^2)^{-1/2} \) if \( N \) is even, and \( H_a(0) = 1 \) if \( N \) is odd. Given the passband and stopband cutoff frequencies, \( \Omega_p \) and \( \Omega_s \), and the passband and stopband ripples, \( \delta_p \) and \( \delta_s \) (or the parameters \( \epsilon \) and \( A \)), the steps involved in designing a type I Chebyshev filter are as follows:

1. Find the values for the selectivity factor, \( k \), and the discrimination factor, \( d \).
2. Determine the filter order using the formula

\[ N \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} \]

3. Form the rational function

\[ G_a(s) = H_a(s)H_a(-s) = \frac{1}{1 + \epsilon^2 T_N^2(s/j\Omega_p)} \]

where \( \epsilon = [(1 - \delta_p)^{-2} - 1]^{1/2} \), and construct the system function \( H_a(s) \) by taking the \( N \) poles of \( G_a(s) \) that lie in the left-half \( s \)-plane.

**EXAMPLE 9.4.2** If we were to design a low-pass type I Chebyshev filter to meet the specifications given in Example 9.4.1 where we found \( d = 0.0487 \) and \( k = 0.6 \), the required filter order would be

\[ N \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = 3.38 \]

or \( N = 4 \). Therefore, with

\[ \epsilon = [(1 - \delta_p)^{-2} - 1]^{1/2} = 0.4843 \]

and

\[ T_4(x) = 4x^3 - 4x \]

then

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + 3.7527(\Omega/\Omega_p)^2[(\Omega/\Omega_p)^2 - 1]^2} \]

where \( \Omega_p = 2\pi(6000) \).
A type II Chebyshev filter, unlike a type I filter, has a monotonic passband and an equiripple stopband, and the system function has both poles and zeros. The magnitude of the frequency response is

$$|H_0(j\Omega)|^2 = \frac{1}{1 + \epsilon^2(T_N(\Omega_s/\Omega_p)/T_N(\Omega_s/\Omega))^2}$$

where \(N\) is the order of the filter, \(\Omega_p\) is the passband cutoff frequency, \(\Omega_s\) is the stopband cutoff frequency, and \(\epsilon\) is the parameter that controls the stopband ripple amplitude. Again, as the order \(N\) is increased, the number of ripples increases and the transition width becomes narrower. Examples are given in Fig. 9-9 for \(N = 5, 6\).

![Frequency response of a Chebyshev type II filter for orders \(N = 5\) and \(N = 6\).](image)

The system function of a type II Chebyshev filter has the form

$$H_0(s) = \prod_{k=0}^{N-1} \frac{\Omega_s^2}{b_k s - a_k}$$

The poles are located at

$$a_k = \frac{\Omega_s^2}{s_k}$$

where \(s_k\) for \(k = 0, 1, \ldots, N - 1\) are the poles of a type I Chebyshev filter. The zeros \(b_k\) lie on the \(j\Omega\)-axis at the frequencies for which \(T_N(\Omega_s/\Omega) = 0\). The procedure for designing a type II Chebyshev filter is the same as for a type I filter, except that

$$\epsilon = (\delta_s^2 - 1)^{-1/2}$$

**Elliptic Filter**

An elliptic filter has a system function with both poles and zeros. The magnitude of its frequency response is

$$|H_0(\Omega)|^2 = \frac{1}{1 + \epsilon^2 U_N^2(\Omega/\Omega_p)}$$

where \(U_N(\Omega/\Omega_p)\) is a Jacobian elliptic function. The Jacobian elliptic function \(U_N(x)\) is a rational function of order \(N\) with the following property:

$$U_N\left(\frac{1}{\Omega}\right) = \frac{1}{U_N(\Omega)}$$

Elliptic filters have an equiripple passband and an equiripple stopband. Because the ripples are distributed uniformly across both bands (unlike the Butterworth and Chebyshev filters, which have a monotonically decreasing
passband and/or stopband), these filters are optimum in the sense of having the smallest transition width for a given filter order, cutoff frequency $\Omega_c$, and passband and stopband ripples. The frequency response for a 4th-order elliptic filter is shown in Fig. 9-10.

$$|H_a(j\Omega)|^2$$

![Graph of frequency response](image)

Fig. 9-10. The magnitude of the frequency response of a sixth-order elliptic filter.

The design of elliptic filters is more difficult than the Butterworth and Chebyshev filters, because their design relies on the use of tables or series expansions. However, the filter order necessary to meet a given set of specifications may be estimated using the formula

$$N \geq \frac{\log(16/d^2)}{\log(1/q)}$$

where $d$ is the discrimination factor, and

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13}$$

where

$$q_0 = \frac{11 - (1 - k^2)^{1/4}}{2(1 + (1-k^2)^{1/4})}$$

with $k$ being the selectivity factor.

### 9.4.2 Design of IIR Filters from Analog Filters

The design of a digital filter from an analog prototype requires that we transform $h_a(t)$ to $h(n)$ or $H_a(s)$ to $H(z)$. A mapping from the $s$-plane to the $z$-plane may be written as

$$H(z) = H_a(s)\bigg|_{s = m(z)}$$

where $s = m(z)$ is the mapping function. In order for this transformation to produce an acceptable digital filter, the mapping $m(z)$ should have the following properties:

1. The mapping from the $j\Omega$-axis to the unit circle, $|z| = 1$, should be one to one and onto the unit circle in order to preserve the frequency response characteristics of the analog filter.
2. Points in the left-half $s$-plane should map to points inside the unit circle to preserve the stability of the analog filter.
3. The mapping $m(z)$ should be a rational function of $z$ so that a rational $H_a(s)$ is mapped to a rational $H(z)$.

Described below are two approaches that are commonly used to map analog filters into digital filters.
Impulse Invariance

With the *impulse invariance* method, a digital filter is designed by sampling the impulse response of an analog filter:

\[ h(n) = h_a(nT_s) \]

From the sampling theorem, it follows that the frequency response of the digital filter, \( H(e^{j\omega}) \), is related to the frequency response \( H_a(j\Omega) \) of the analog filter as follows:

\[ H(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H_a \left( \frac{j\omega}{T_s} + j \frac{2\pi k}{T_s} \right) \]

More generally, this may be extended into the complex plane as follows:

\[ H(z) \big|_{z=e^{j\omega}} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H_a \left( s + j \frac{2\pi k}{T_s} \right) \]

The mapping between the s-plane and the z-plane is illustrated in Fig. 9-11. Note that although the \( j\Omega \)-axis maps onto the unit circle, the mapping is not one to one. In particular, each interval of length \( 2\pi / T_s \) along the \( j\Omega \)-axis is mapped onto the unit circle (i.e., the frequency response is aliased). In addition, each point in the left-half s-plane is mapped to a point inside the unit circle. Specifically, strips of width \( 2\pi / T_s \) map onto the z-plane. If the frequency response of the analog filter, \( H_a(j\Omega) \), is sufficiently bandlimited, then

\[ H(e^{j\omega}) \approx \frac{1}{T_s} H_a \left( \frac{j\omega}{T_s} \right) \]

Although the impulse invariance may produce a reasonable design in some cases, this technique is essentially limited to bandlimited analog filters.

![Fig. 9-11. Properties of the s-plane to z-plane mapping in the impulse invariance method.](image)

To see how poles and zeros of an analog filter are mapped using the impulse invariance method, consider an analog filter that has a system function

\[ H_a(s) = \sum_{k=1}^{\rho} \frac{A_k}{s - s_k} \]

The impulse response, \( h_a(t) \), is

\[ h_a(t) = \sum_{k=1}^{\rho} A_k e^{s_k t} u(t) \]
Therefore, the digital filter that is formed using the impulse invariance technique is
\[ h(n) = h_d(nT_s) = \sum_{k=1}^{p} A_k e^{\frac{n}{T_s}} u(n) = \sum_{k=1}^{p} A_k (e^{\frac{nT_s}{T_s}})^k u(n) \]
and the system function is
\[ H(z) = \sum_{k=1}^{p} \frac{A_k}{1 - e^{\frac{2\pi k}{T_s} T_s}} z^{-1} \quad (9.10) \]
Thus, a pole at \( s = s_k \) in the analog filter is mapped to a pole at \( z = e^{\frac{T_s}{T_s}} \) in the digital filter.
\[ \frac{1}{s - s_k} \Rightarrow \frac{1}{1 - e^{\frac{T_s}{T_s}} z^{-1}} \]
The zeros, however, do not get mapped in any obvious way.

The Bilinear Transformation

The bilinear transformation is a mapping from the \( s \)-plane to the \( z \)-plane defined by
\[ s = \frac{s}{T_s} = \frac{2}{1 + z^{-1}} \quad (9.11) \]
Given an analog filter with a system function \( H_o(s) \), the digital filter is designed as follows:
\[ H(z) = H_o \left( \frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}} \right) \]
The bilinear transformation is a rational function that maps the left-half \( s \)-plane inside the unit circle and maps the \( j\Omega \)-axis in a one-to-one manner onto the unit circle. However, the relationship between the \( j\Omega \)-axis and the unit circle is highly nonlinear and is given by the frequency warping function
\[ \omega = 2 \arctan \left( \frac{\Omega T_s}{2} \right) \quad (9.12) \]
As a result of this warping, the bilinear transformation will only preserve the magnitude response of analog filters that have an ideal response that is piecewise constant. Therefore, the bilinear transformation is generally only used in the design of frequency selective filters.

The parameter \( T_s \) in the bilinear transformation is normally included for historical reasons. However, it does not enter into the design process, because it only scales the \( j\Omega \)-axis in the frequency warping function, and this scaling may be done in the specification of the analog filter. Therefore, \( T_s \) may be set to any value to simplify the design procedure. The steps involved in the design of a digital low-pass filter with a passband cutoff frequency \( \omega_p \), stopband cutoff frequency \( \omega_s \), passband ripple \( \delta_p \), and stopband ripple \( \delta_s \), are as follows:

1. **Pre Warp** the passband and stopband cutoff frequencies of the digital filter, \( \omega_p \) and \( \omega_s \), using the inverse of Eq. (9.12) to determine the passband and cutoff frequencies of the analog low-pass filter. With \( T_s = 2 \), the prewarping function is
\[ \Omega = \tan \left( \frac{\omega}{2} \right) \]
2. Design an analog low-pass filter with the cutoff frequencies found in step 1 and a passband and stopband ripple \( \delta_p \) and \( \delta_s \), respectively.
3. Apply the bilinear transformation to the filter designed in step 2.

**EXAMPLE 9.4.3** Let us design a first-order digital low-pass filter with a 3-dB cutoff frequency of \( \omega_c = 0.25\pi \) by applying the bilinear transformation to the analog Butterworth filter
\[ H_o(s) = \frac{1}{1 + s/\Omega} \]
Because the 3-dB cutoff frequency of the Butterworth filter is \( \Omega_c \), for a cutoff frequency \( \omega_c = 0.25\pi \) in the digital filter, we must have

\[
\Omega_c = \frac{2}{T_s} \tan \left( \frac{0.25\pi}{2} \right) = \frac{0.828}{T_s}
\]

Therefore, the system function of the analog filter is

\[
H_a(s) = \frac{1}{1 + sT_s/0.828}
\]

Applying the bilinear transformation to the analog filter gives

\[
H(z) = H_a(s) \bigg|_{s = \frac{1}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}}} = \frac{1}{1 + (2/0.828)(1 - z^{-1})/(1 + z^{-1})} = 0.2920 \frac{1 + z^{-1}}{1 - 0.4159z^{-1}}
\]

Note that the parameter \( T_s \) does not enter into the design.

### 9.4.3 Frequency Transformations

The preceding section considered the design of digital low-pass filters from analog low-pass filters. There are two approaches that may be used to design other types of frequency selective filters, such as high-pass, bandpass, or bandstop filters. The first is to design an analog low-pass filter and then apply a frequency transformation to map the analog filter into the desired frequency selective prototype. This analog prototype is then mapped to a digital filter using a suitable \( s \)-plane to \( z \)-plane mapping. Table 9-5 provides a list of some analog-to-analog transformations.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Mapping</th>
<th>New Cutoff Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass</td>
<td>( s \rightarrow \frac{\Omega_p}{\Omega_c} s )</td>
<td>( \Omega'_p )</td>
</tr>
<tr>
<td>High-pass</td>
<td>( s \rightarrow \frac{\Omega_p \Omega'_p}{s} )</td>
<td>( \Omega'_p )</td>
</tr>
<tr>
<td>Bandpass</td>
<td>( s \rightarrow \frac{s^2 + \Omega_p \Omega'_p}{s(\Omega_p - \Omega_c)} )</td>
<td>( \Omega_p, \Omega_c )</td>
</tr>
<tr>
<td>Bandstop</td>
<td>( s \rightarrow \frac{s(\Omega_p - \Omega_c)}{s^2 + \Omega_p \Omega'_p} )</td>
<td>( \Omega_p, \Omega_c )</td>
</tr>
</tbody>
</table>

The second approach that may be used is to design an analog low-pass filter, map it into a digital filter using a suitable \( s \)-plane to \( z \)-plane mapping, and then apply an appropriate frequency transformation in the discrete-time domain to produce the desired frequency selective digital filter. Table 9-6 provides a list of some digital-to-digital transformations. The two approaches do not always result in the same design. For example, although the second approach could be used to design a high-pass filter using the impulse invariance technique, with the first approach the design would be unacceptable due to the aliasing that would occur when sampling the analog high-pass filter.

### 9.5 FILTER DESIGN BASED ON A LEAST SQUARES APPROACH

The design techniques described in the previous section are based on converting an analog filter into a digital filter. It is also possible to perform the design directly in the time domain without any reference to an analog filter. This section describes several methods for designing a digital filter directly.
Table 9-6  The Transformation of a Digital Low-Pass Filter with a Cutoff Frequency \( \omega_c \) to Other Frequency Selective Filters

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Mapping</th>
<th>Design Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass</td>
<td>( z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} )</td>
<td>( \alpha = \frac{\sin(\omega_c - \omega'_c)/2}{\sin(\omega_c + \omega'_c)/2} ) ( \omega'_c ) desired cutoff frequency</td>
</tr>
<tr>
<td>High-pass</td>
<td>( z^{-1} \rightarrow \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}} )</td>
<td>( \alpha = -\frac{\cos(\omega_c + \omega'_c)/2}{\cos(\omega_c - \omega'_c)/2} ) ( \omega'_c ) desired cutoff frequency</td>
</tr>
<tr>
<td>Bandpass</td>
<td>( z^{-1} \rightarrow \frac{z^{-2} - [2\alpha\beta/(\beta + 1)]z^{-1} + ([\beta - 1]/(\beta + 1))z^{-2} - [2\alpha\beta/(\beta + 1)]z^{-1} + 1}{[1 - \beta]/(1 + \beta)z^{-2} - [2\alpha/(\beta + 1)]z^{-1} + 1} )</td>
<td>( \alpha = \cos((\omega_c + \omega_{1,2})/2) ) ( \beta = \cot((\omega_{2,1} - \omega_c)/2) \tan(\omega_c/2) ) ( \omega_{1,2} ) desired upper cutoff frequency</td>
</tr>
<tr>
<td>Bandstop</td>
<td>( z^{-1} \rightarrow \frac{z^{-2} - [2\alpha/(\beta + 1)]z^{-1} + [(1 - \beta)/(1 + \beta)]z^{-2} - [2\alpha/(\beta + 1)]z^{-1} + 1}{[1 - \beta]/(1 + \beta)z^{-2} - [2\alpha/(\beta + 1)]z^{-1} + 1} )</td>
<td>( \alpha = \cos((\omega_{1,2} + \omega_{1,2})/2) ) ( \beta = \tan((\omega_{2,1} - \omega_{1,2})/2) \tan(\omega_c/2) ) ( \omega_{1,2} ) desired lower cutoff frequency</td>
</tr>
</tbody>
</table>

### 9.5.1 Padé Approximation

Let \( h_d(n) \) be the unit sample response of an ideal filter that is to be approximated by a causal filter that has a unit sample response, \( h(n) \), and a rational system function, \( H(z) \).

\[
H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} = \frac{\sum_{k=0}^{p} b(k)z^{-k}}{1 + \sum_{k=1}^{p} a(k)z^{-k}} \tag{9.13}
\]

Because \( H(z) \) has \( p + q + 1 \) free parameters, it is generally possible to find values for the coefficients \( a(k) \) and \( b(k) \) so that \( h(n) = h_d(n) \) for \( n = 0, 1, \ldots, p + q \). The procedure that is used to find these coefficients is to write \( H(z) = B(z)/A(z) \) as follows,

\[
A(z)H(z) = B(z)
\]

and note that, in the time domain, the left-hand side corresponds to a convolution

\[
a(n) \ast h(n) = h(n) + \sum_{k=1}^{p} a(k)h(n-k) = b(n)
\]

(note that \( b(n) \) is a finite-length sequence that is equal to zero for \( n < 0 \) and \( n > q \)). Setting \( h(n) = h_d(n) \) for \( n = 0, 1, \ldots, p + q \) results in a set of \( p + q + 1 \) linear equations in \( p + q + 1 \) unknowns,

\[
h_d(n) + \sum_{k=1}^{p} a(k)h_d(n-k) = \begin{cases} b(n) & n = 0, 1, \ldots, q \\ 0 & n = q + 1, \ldots, q + p \end{cases} \tag{9.14}
\]
that may be solved using a two-step approach. In the first step, the coefficients \( a(k) \) are found using the last \( p \) equations in Eq. (9.14), which may be written in matrix form as

\[
\begin{bmatrix}
    h_d(q) & h_d(q - 1) & \cdots & h_d(q - p + 1) \\
    h_d(q + 1) & h_d(q) & \cdots & h_d(q - p + 2) \\
    \vdots & \vdots & \ddots & \vdots \\
    h_d(q + p - 1) & h_d(q + p - 2) & \cdots & h_d(q)
\end{bmatrix}
\begin{bmatrix}
    a(1) \\
    a(2) \\
    \vdots \\
    a(p)
\end{bmatrix}
= -
\begin{bmatrix}
    h_d(q + 1) \\
    h_d(q + 2) \\
    \vdots \\
    h_d(q + p)
\end{bmatrix}
\]

Assuming that these equations are linearly independent, the coefficients may be uniquely determined. In the second step, the coefficients \( b(k) \) are found from the first \( q + 1 \) equations in Eq. (9.14) as follows:

\[
b(n) = h_d(n) + \sum_{k=1}^{p} a(k) h_d(n - k) \quad n = 0, 1, \ldots, q
\]

Although Padé’s method produces an exact match of \( h(n) \) to \( h_d(n) \) for \( n = 0, 1, \ldots, p + q \), because \( h(n) \) is unconstrained for \( n > p + q \), the Padé method does not generally produce a good approximation to \( h_d(n) \) for \( n > p + q \).

### 9.5.2 Prony’s Method

With a least-squares approach to filter design, the problem is to find the coefficients \( a(k) \) and \( b(k) \) that minimize the least-squares error

\[
\mathcal{E} = \sum_{n=0}^{U} |h_d(n) - h(n)|^2
\]

(9.15)

where \( U \) is some preselected upper limit. Because \( \mathcal{E} \) is a nonlinear function of the coefficients \( a(k) \) and \( b(k) \), solving this minimization problem is, in general, difficult. With Prony’s method, however, an approximate least-squares solution may be found using a two-step procedure as follows. Ideally, because [see Eq. (9.14)]

\[
h_d(n) + \sum_{k=1}^{p} a(k) h_d(n - k) = 0 \quad n \geq q + 1
\]

the first step is to find the coefficients \( a(k) \) that minimize

\[
\mathcal{E} = \sum_{n=q+1}^{\infty} e^2(n)
\]

where

\[
e(n) = h_d(n) + \sum_{k=1}^{p} a(k) h_d(n - k)
\]

Once the coefficients \( a(k) \) have been determined, the coefficients \( b(k) \) are found using the Padé approach of forcing \( h(n) = h_d(n) \) for \( n = 0, 1, \ldots, q \):

\[
b(n) = \sum_{k=1}^{p} a(k) h_d(n - k) \quad 0 \leq n \leq q
\]

The coefficients \( a(k) \) that minimize \( \mathcal{E} \) may be found by setting the partial derivatives of \( \mathcal{E} \) equal to zero,

\[
\frac{\partial \mathcal{E}}{\partial a(k)} = 0 \quad k = 0, 1, \ldots, p
\]
and solving for the unknowns $a(k)$. Setting the derivatives equal to zero produces the following set of linear equations:

$$
\begin{bmatrix}
    r_d(1, 1) & r_d(1, 2) & \cdots & r_d(1, p) \\
    r_d(2, 1) & r_d(2, 2) & \cdots & r_d(2, p) \\
    \vdots & \vdots & \ddots & \vdots \\
    r_d(p, 1) & r_d(p, 2) & \cdots & r_d(p, p)
\end{bmatrix}
\begin{bmatrix}
    a(1) \\
    a(2) \\
    \vdots \\
    a(p)
\end{bmatrix}
= -
\begin{bmatrix}
    r_d(1, 0) \\
    r_d(2, 0) \\
    \vdots \\
    r_d(p, 0)
\end{bmatrix}
$$

(9.16)

where

$$
r_d(k, l) = \sum_{n=\infty}^{\infty} h_d(n-l)h_d(n-k)
$$

is the correlation of $h_d(n)$.

### 9.5.3 FIR Least-Squares Inverse

The inverse of a linear shift-invariant system with unit sample response $g(n)$ and system function $G(z)$ is the system that has a unit sample response, $h(n)$, such that

$$
h(n) \ast g(n) = \delta(n)
$$

or

$$
H(z)G(z) = 1
$$

In most applications, the system function $H(z) = 1/G(z)$ is not a viable solution. One of the reasons is that, unless $G(z)$ is minimum phase, $1/G(z)$ cannot be both causal and stable. Another consideration comes from the fact that, in some applications, it may be necessary to constrain $H(z)$ to be an FIR filter. Because $1/G(z)$ will be infinite in length unless $G(z)$ is an all-pole filter, constraining $h(n)$ to be FIR would only be an approximation to the inverse filter.

In the FIR least-squares inverse filter design problem, the goal is to find the FIR filter $h(n)$ of length $N$ such that

$$
h(n) \ast g(n) \approx \delta(n)
$$

The filter that minimizes the squared error

$$
E = \sum_{n=0}^{\infty} |e(n)|^2
$$

where

$$
e(n) = \delta(n) - h(n) \ast g(n) = \delta(n) - \sum_{l=0}^{N-1} h(l)g(n-l)
$$

(9.17)

may be found by solving the linear equations

$$
\sum_{l=0}^{N-1} h(l)r_g(k-l) = \begin{cases} 
    g(0) & k = 0 \\
    0 & k = 1, 2, \ldots, N-1
\end{cases}
$$

(9.18)

where

$$
r_g(k) = \sum_{n=0}^{\infty} g(n)g(n-k)
$$

In many cases, constraining the least-squares inverse filter to minimize the difference between $h(n) \ast g(n)$ and $\delta(n)$ is overly restrictive. For example, if a delay may be tolerated, we may consider finding the filter $h(n)$ so that

$$
h(n) \ast g(n) \approx \delta(n - n_0)
$$

for some delay $n_0$. In most cases, a nonzero delay will produce a better approximate inverse filter and, in many cases, the improvement will be substantial. The least-squares inverse filter with delay is found by solving the linear equations

$$
\sum_{l=0}^{N-1} h(l)r_g(k-l) = \begin{cases} 
    g(n_0-k) & k = 0, 1, \ldots, n_0 \\
    0 & k = n_0 + 1, \ldots, N
\end{cases}
$$

(9.19)
Solved Problems

FIR Filter Design

9.1 Use the window design method to design a linear phase FIR filter of order \( N = 24 \) to approximate the following ideal frequency response magnitude:

\[
|H_e(j\omega)| = \begin{cases} 
1 & |\omega| \leq 0.2\pi \\
0 & 0.2\pi < |\omega| \leq \pi 
\end{cases}
\]

The ideal filter that we would like to approximate is a low-pass filter with a cutoff frequency \( \omega_c = 0.2\pi \). With \( N = 24 \), the frequency response of the filter that is to be designed has the form

\[
H(e^{j\omega}) = \sum_{n=0}^{24} h(n)e^{-j\omega n}
\]

Therefore, the delay of \( h(n) \) is \( \alpha = N/2 = 12 \), and the ideal unit sample response that is to be windowed is

\[
h_d(n) = \frac{\sin[0.2\pi(n-12)]}{(n-12)\pi}
\]

All that is left to do in the design is to select a window. With the length of the window fixed, there is a trade-off between the width of the transition band and the amplitude of the passband and stopband ripple. With a rectangular window, which provides the smallest transition band,

\[
h(n) = \begin{cases} 
\sin[0.2\pi(n-12)] & 0 \leq n \leq 24 \\
0 & \text{otherwise}
\end{cases}
\]

However, the stopband attenuation is only 21 dB, which is equivalent to a ripple of \( \delta_s = 0.089 \). With a Hamming window, on the other hand,

\[
h(n) = \left[0.54 - 0.46 \cos \left(\frac{2\pi n}{24}\right)\right] \cdot \frac{\sin[0.2\pi(n-12)]}{(n-12)\pi} \quad 0 \leq n \leq 24
\]

and the stopband attenuation is 53 dB, or \( \delta_s = 0.0022 \). However, the width of the transition band increases to

\[
\Delta \omega = 2\pi \cdot \frac{3.3}{24} = 0.275\pi
\]

which, for most designs, would be too wide.

9.2 Use the window design method to design a minimum-order high-pass filter with a stopband cutoff frequency \( \omega_s = 0.22\pi \), a passband cutoff frequency \( \omega_p = 0.28\pi \), and a stopband ripple \( \delta_s = 0.003 \).

A stopband ripple of \( \delta_s = 0.003 \) corresponds to a stopband attenuation of \( \alpha_s = -20 \log \delta_s = 50.46 \). For the minimum-order filter, we use a Kaiser window with

\[
\beta = 0.1102(\alpha_s - 8.7) = 4.6
\]

Because the transition width is \( \Delta \omega = 0.06\pi \), or \( \Delta f = 0.03 \), the required window length is

\[
N = \frac{\alpha_s - 7.95}{14.36 \Delta f} = 98.67
\]
Rounding this up to \( N = 99 \) results in a type II linear phase filter, which will have a zero in its system function at \( z = -1 \). Because this produces a null in the frequency response at \( \omega = \pi \), this is not acceptable. Therefore, we increase the order by 1 to obtain a type I linear phase filter with \( N = 100 \).

In order to have a transition band that extends from \( \omega_L = 0.22\pi \) to \( \omega_R = 0.28\pi \), we set the cutoff frequency of the ideal high-pass filter equal to the midpoint:

\[
\omega_c = \frac{\omega_L + \omega_R}{2} = 0.25\pi
\]

The unit sample response of an ideal zero-phase high-pass filter with a cutoff frequency \( \omega_c = 0.25\pi \) is

\[
h_{hp}(n) = \delta(n) - \frac{\sin(0.25\pi n)}{n\pi}
\]

where the second term is a low-pass filter with a cutoff frequency \( \omega_c = 0.25\pi \). Delaying \( h_{hp}(n) \) by \( N/2 = 50 \), we have

\[
h_d(n) = \delta(n - 50) - \frac{\sin[0.25\pi(n - 50)]}{(n - 50)\pi}
\]

and the resulting FIR high-pass filter is

\[
h(n) = h_d(n) \cdot w(n)
\]

where \( w(n) \) is a Kaiser window with \( N = 100 \) and \( \beta = 4.6 \).

### 9.3
Given a desired frequency response \( H_d(e^{j\omega}) \), show that the rectangular window design minimizes the least-squares error

\[
E_{LS} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 d\omega
\]

For this problem, we use Parseval’s theorem to express the least-squares error \( E_{LS} \) in the time domain:

\[
E_{LS} = \sum_{n=-\infty}^{\infty} |h_d(n) - h(n)|^2
\]

If we assume that \( h(n) \) is of order \( N \), with \( h(n) = 0 \) for \( n < 0 \) and \( n > N \),

\[
E_{LS} = \sum_{n=0}^{N} |h_d(n) - h(n)|^2 + \sum_{n=-\infty}^{-1} |h_d(n)|^2 + \sum_{n=N+1}^{\infty} |h_d(n)|^2
\]

Because the last two terms are constants that are not affected by the filter \( h(n) \), the least-squares error \( E_{LS} \) is minimized by minimizing the first term, which is done by setting \( h(n) = h_d(n) \) for \( n = 0, 1, \ldots, N \) (i.e., using a rectangular window in the window design method).

### 9.4
If \( h_d(n) \) is the unit sample response of an ideal filter, and \( h(n) \) is an \( N \)th-order FIR filter, the least-squares error

\[
E_{LS} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 d\omega
\]

is minimized when \( h(n) \) is designed using the rectangular window design method. If \( E_R \) is the squared error using a rectangular window, find the excess squared error that results when a Hanning window is used instead of a rectangular window; that is, find an expression for

\[
E_{ex} = E_{H} - E_R
\]

where \( E_H \) is the squared error using a Hanning window.

Using Parseval’s theorem, it is more convenient to express the least-squares error in the time domain as follows:

\[
E_{LS} = \sum_{n=-\infty}^{\infty} |h_d(n) - h(n)|^2
\]
Because \( e(n) = h_d(n) - h(n) \) for \( n < 0 \) and \( n > N \),

\[
E_{ex} = E_{ll} - E_R = \sum_{n=0}^{N} |h_d(n) - w_H(n)h_d(n)|^2 - \sum_{n=0}^{N} |h_d(n) - w_R(n)h_d(n)|^2
\]

where \( w_H(n) \) and \( w_R(n) \) are the Hanning and rectangular windows, respectively. However, the second sum is equal to zero. Therefore, the excess squared error is simply

\[
E_{ex} = \sum_{n=0}^{N} |h_d(n) - w_H(n)h_d(n)|^2 = \sum_{n=0}^{N} |1 - w_H(n)|^2 |h_d(n)|^2 = 0.5 + 0.5 \cos\left(\frac{2\pi n}{N}\right)^2 |h_d(n)|^2
\]

which is the desired relationship.

9.5 Consider the following specifications for a low-pass filter:

\[
0.99 \leq |H(e^{j\omega})| \leq 1.01 \quad 0 \leq |\omega| \leq 0.3\pi
\]

\[
|H(e^{j\omega})| \leq 0.01 \quad 0.35\pi \leq |\omega| \leq \pi
\]

Design a linear phase FIR filter to meet these specifications using the window design method.

Designing a low-pass filter with the window design method generally produces a filter with ripples of the same amplitude in the passband and stopband. Therefore, because the passband and stopband ripples in the filter specifications are the same, we only need to be concerned about the stopband ripple requirement. A stopband ripple of \( \delta_v = 0.01 \) corresponds to a stopband attenuation of \(-40 \text{ dB}\). Therefore, from Table 9-2 it follows that we may use a Hanning window, which provides an attenuation of approximately \(44 \text{ dB}\). The specification on the transition band is that \( \Delta \omega = 0.05\pi \), or \( \Delta f = 0.025 \). Therefore, the required filter order is

\[
N = \frac{3.1}{\Delta f} = 124
\]

and we have

\[
w(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{124}\right) \quad 0 \leq n \leq 124
\]

With an ideal low-pass filter that has a cutoff frequency of \( \omega_c = 0.325 \) (the midpoint of the transition band), and a delay of \( N/2 = 62 \) so that \( h_d(n) \) is placed symmetrically within the interval \([0, 124]\), we have

\[
h_d(n) = \frac{\sin[0.325\pi(n - 62)]}{\pi(n - 62)}
\]

Therefore, the filter is

\[
h(n) = \left[0.5 - 0.5 \cos\left(\frac{2\pi n}{124}\right)\right] \cdot \frac{\sin[0.325\pi(n - 62)]}{\pi(n - 62)} \quad 0 \leq n \leq 124
\]

Note that if we were to use a Hamming or a Blackman window instead of a Hanning window, the stopband and passband ripple requirements would have been exceeded, and the required filter order would have been larger. With a Blackman window, for example, the filter order required to meet the transition band requirement is

\[
N = \frac{5.5}{0.025} = 220
\]

9.6 We would like to filter an analog signal \( x_a(t) \) with an analog low-pass filter that has a cutoff frequency \( f_c = 2 \text{ kHz} \), a transition width \( \Delta f = 500 \text{ Hz} \), and a stopband attenuation of \(50 \text{ dB}\). This filter is to be implemented digitally, as illustrated in the following figure:
Design a digital filter to meet the analog filter specifications with a sampling frequency $f_s = 10$ kHz.

With a sampling frequency of 10 kHz, the digital filter should have a cutoff frequency $\omega_c = 2\pi f_s/f_s = 0.4\pi$ and a transition bandwidth $\Delta \omega = 2\pi \Delta f/f_s = 0.1\pi$. For a stopband attenuation of 50 dB, we may use a Kaiser window with

$$\beta = 0.1102(50 - 8.7) = 4.55$$

For the length of the window, we have

$$N = \frac{-20 \log(\delta_s) - 7.95}{14.36 \cdot \Delta f} = \frac{50 - 7.95}{14.36 \cdot 0.05} = 58.56$$

or $N = 59$. Finally, the unit sample response of the ideal filter that is to be windowed is a low-pass filter with a cutoff frequency $\omega_c = 0.4\pi$ and a delay $N/2 = 29.5$. Therefore,

$$h(n) = w(n) h_d(n)$$

where $w(n)$ is a Kaiser window with $N = 59$ and $\beta = 4.55$, and

$$h_d(n) = \frac{\sin(0.4\pi(n - 29.5))}{(n - 29.5)\pi}$$

9.7 Find the Kaiser window parameters, $\beta$ and $N$, to design a low-pass filter with a cutoff frequency $\omega_c = \pi/2$, a stopband ripple $\delta_s = 0.002$, and a transition bandwidth no larger than $0.1\pi$.

The parameter $\beta$ for the Kaiser window depends only on the stopband ripple requirements. With $\delta_s = 0.002$,

$$\alpha_s = -20 \log(0.002) = 53.98$$

and we have

$$\beta = 0.1102(\alpha_s - 8.7) = 4.99$$

The window length, $N$, on the other hand, is determined by the stopband ripple, $\delta_s$, and the transition width as follows:

$$N = \frac{\alpha_s - 7.95}{14.36 \Delta f} = 64.1$$

Therefore, the required filter order is $N = 65$.

9.8 Consider the following specifications for a bandpass filter:

$$|H(e^{j\omega})| \leq 0.01 \quad 0 \leq |\omega| \leq 0.2\pi$$

$$0.95 \leq |H(e^{j\omega})| \leq 1.05 \quad 0.3\pi \leq |\omega| \leq 0.7\pi$$

$$|H(e^{j\omega})| \leq 0.02 \quad 0.8\pi \leq |\omega| \leq \pi$$

(a) Design a linear phase FIR filter to meet these specifications using a Blackman window.

(b) Repeat part (a) using a Kaiser window.

(a) For this filter, the width of each transition band is $\Delta \omega = 0.1\pi$. The ripples in the lower stopband, passband, and upper stopband are $\delta_1 = 0.01$, $\delta_2 = 0.05$, and $\delta_3 = 0.02$, respectively, and are all different. Because the ripples produced with the window design method will be approximately the same in all three bands, the filter must be designed so that it has a maximum ripple of $\delta_1 = 0.01$ in all three bands. With

$$-20 \log \delta_1 = -40$$
it follows that the Blackman window will satisfy this requirement. An estimate of the filter order necessary to meet the transition bandwidth requirement of $\Delta f = 0.05$ with a Blackman window is

$$N = \frac{5.5}{\Delta f} = 110$$

Finally, for the unit sample response of the ideal filter that is to be windowed, we have

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n} d\omega$$

where $H_d(e^{j\omega})$ is the frequency response of an ideal bandpass filter. For the cutoff frequencies of $H_d(e^{j\omega})$, we choose the midpoints of the transition bands of $H(e^{j\omega})$. Therefore,

$$|H_d(e^{j\omega})| = \begin{cases} 1 & 0.25\pi \leq |\omega| \leq 0.75\pi \\ 0 & \text{otherwise} \end{cases}$$

Thus, the unit sample response of the ideal bandpass filter with zero phase is

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-0.25\pi}^{-0.75\pi} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{0.25\pi}^{0.75\pi} e^{j\omega n} d\omega$$

and

$$h_d(n) = \frac{1}{2j\pi} \left[ e^{-j0.25\pi n} - e^{-j0.75\pi n} + e^{j0.75\pi n} - e^{j0.25\pi n} \right] = \frac{\sin(0.75\pi n)}{n\pi} - \frac{\sin(0.25\pi n)}{n\pi}$$

However, we want to delay this filter so that it is centered at $N/2 = 55$. Therefore, the unit sample response of the filter that is to be windowed should be

$$h_d(n) = \frac{\sin[0.75\pi(n - 55)]}{(n - 55)\pi} - \frac{\sin[0.25\pi(n - 55)]}{(n - 55)\pi}$$

(b) For a Kaiser window design, the order of the filter that is required is

$$N = -\frac{20 \log(0.01) - 7.95}{14.36(0.05)} = 44.64$$

Therefore, we set $N = 45$. Next, for the Kaiser window parameter, with an attenuation of 40 dB, we have

$$\beta = 0.5842(40 - 21)^{0.4} + 0.07886(40 - 21) = 3.3953$$

Therefore, the filter is

$$h(n) = w(n) \cdot h_d(n)$$

where

$$h_d(n) = \frac{\sin[0.75\pi(n - 22.5)]}{(n - 22.5)\pi} - \frac{\sin[0.25\pi(n - 22.5)]}{(n - 22.5)\pi}$$

9.9 Suppose that we would like to design a bandstop filter to meet the following specifications:

$$0.95 \leq |H(e^{j\omega})| \leq 1.05 \quad 0 \leq |\omega| \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.005 \quad 0.22\pi \leq |\omega| \leq 0.75\pi$$

$$0.95 \leq |H(e^{j\omega})| \leq 1.05 \quad 0.8\pi \leq |\omega| \leq \pi$$
(a) Design a linear phase FIR filter to meet these filter specifications using the window design method.

(b) What is the approximate order of the equiripple filter that will meet these specifications?

(a) Recall that with the window design method, the ripples in the passbands and stopbands will be approximately the same, along with the widths of the transition bands. Because the smallest ripple occurs in the stopband, we must pick a window that provides a stopband attenuation of

\[ \alpha_s = -20 \log(0.005) = 46.02 \]

Thus, we may use a Hamming window or a Kaiser window with

\[ \beta = 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) = 4.09 \]

The transition width between the lower stopband and the passband is \( \Delta \omega = 0.02\pi \) and between the upper stopband and the passband it is \( \Delta \omega = 0.05\pi \). Therefore, we must design the filter to meet the lower transition bandwidth requirement, \( \Delta \omega = 0.02\pi \), or \( \Delta f = 0.01 \). Thus, for a Hamming window, the estimated filter order is

\[ N = \frac{3.3}{0.01} = 330 \]

For a Kaiser window, on the other hand, the filter order is

\[ N = \frac{\alpha_s - 7.95}{14.36 \Delta f} = 265.1 \]

or \( N = 266 \).

(b) For an equiripple filter, the filter order may be estimated as follows,

\[ N = \frac{-10 \log(\delta_s \delta_t) - 13}{14.6 \Delta f} = \frac{-10 \log(0.05 \cdot 0.005) - 13}{(14.6)(0.01)} = 157.67 \]

or \( N = 158 \).

9.10 Use the window design method to design a type II bandpass filter according to the following specifications:

\[
\begin{align*}
|H(e^{j\omega})| & \leq 0.0050 & |\omega| & \leq 0.1\pi \\
0.995 & \leq |H(e^{j\omega})| & 0.25\pi & \leq |\omega| & \leq 0.6\pi \\
|H(e^{j\omega})| & \leq 0.0025 & 0.8\pi & \leq |\omega|
\end{align*}
\]

With the window design method, the amplitudes of the ripples in each band of a multiband filter will be approximately equal, and the transition bands will have approximately the same width. Because the requirements on the peak ripple in the three bands of this bandpass filter are not the same, it is necessary to design the filter so that it has the smallest ripple in all three bands, which, in this case, requires that we set \( \delta_s = 0.0025 \). In addition, because the transition bands do not have the same width, it is necessary to set the desired transition width, \( \Delta \omega \), equal to the smaller of the two \( (\Delta \omega = 0.15\pi) \).

With \( \alpha_s = -20 \log \delta_s = 52 \text{ dB} \), it follows that we may use a Hamming window, and with

\[ N \Delta f = 3.3 \]

this gives an estimated filter order of

\[ N = \frac{3.3}{0.075} = 44 \]

For a type II filter, however, \( N \) must be odd, so we set \( N = 45 \).

Now we must find the unit sample response of the ideal bandpass filter that is to be windowed. Because the width of both the upper and lower transition bands will be approximately \( \Delta \omega = 0.15\pi \), for the ideal filter we set the lower cutoff frequency equal to

\[ \omega_1 = 0.25\pi - \frac{\Delta \omega}{2} = 0.175\pi \]
and the upper cutoff frequency equal to

\[ \omega_n = 0.6\pi + \frac{\Delta\omega}{2} = 0.675\pi \]

Therefore, the magnitude of the frequency response of the ideal filter is

\[ |H_0(e^{j\omega})| = \begin{cases} 1 & 0.175\pi \leq |\omega| \leq 0.675\pi \\ 0 & \text{otherwise} \end{cases} \]

Repeating the steps in the derivation of the unit sample response of an ideal bandpass filter given in Prob. 9.8, using the given cutoff frequencies and a delay of \( N/2 = 22.5 \), we have

\[ h_d(n) = \frac{\sin[0.175\pi(n - 22.5)]}{(n - 22.5)\pi} - \frac{\sin[0.675\pi(n - 22.5)]}{(n - 22.5)\pi} \]

9.11 Use the window design method to design a multiband filter that meets the following specifications:

\[
egin{align*}
0.99 & \leq |H(e^{j\omega})| \leq 1.01 & 0 \leq \omega \leq 0.3\pi \\
|H(e^{j\omega})| & \leq 0.01 & 0.35\pi \leq \omega \leq 0.55\pi \\
0.49 & \leq |H(e^{j\omega})| & 0.6\pi \leq \omega \leq \pi 
\end{align*}
\]

To design a multiband filter that meets these specifications using the window design method, we begin by finding the ideal unit sample response. For the frequency response of the ideal filter, we set the cutoff frequencies equal to the midpoint of the transition bands. Therefore, we have

\[ |H_d(e^{j\omega})| = \begin{cases} 1 & |\omega| \leq 0.325\pi \\ 0 & 0.325\pi \leq |\omega| \leq 0.575\pi \\ 0.5 & 0.575\pi \leq |\omega| \leq \pi \end{cases} \]

The unit sample response of this ideal filter may be found easily by noting that \( H_d(e^{j\omega}) \) may be written as an allpass filter with a gain of 0.5 minus a low-pass filter with a gain of 0.5 and a cutoff frequency \( \omega_1 = 0.575\pi \) plus a low-pass filter with a gain of 1 and a cutoff frequency of \( \omega_2 = 0.325\pi \). Therefore, if we assume that \( H_d(e^{j\omega}) \) has linear phase with a delay of \( n_d \),

\[ h_d(n) = 0.5\delta(n) - 0.5\sin[0.575\pi(n - n_d)]/(n - n_d)\pi + \sin[0.325\pi(n - n_d)]/(n - n_d)\pi \]

Having found the ideal unit sample response, the next step is to choose an appropriate window. When \( h_d(n) \) is multiplied by a window \( w(n) \), the frequency response is the convolution of the transform of the window \( W(e^{j\omega}) \) with \( H_d(e^{j\omega}) \). Assuming that the length of the filter is long compared to the inverse of the transition width, so that the discontinuities between the bands may be treated independently, the ripples in the three bands will be approximately the same as they would be for a low-pass filter, except that they will be scaled by the amplitude of the discontinuities at the band edge. Therefore, if the ripple in the lower passband and the stopband are \( \delta_1 \), the ripple in the upper passband will be \( \delta_2/2 \). Consequently, we must use a window that would produce a low-pass filter with a ripple no larger than 0.01. Thus, we may use a Hanning window. Finally, to determine the filter order, note that because the widths of both transition bands are the same, \( \Delta\omega = 0.05\pi \), an estimate of the filter order is

\[ N = \frac{3.1}{\Delta f} = 124 \]

Note that another way to design this filter would have been to design a network of three filters in parallel: a low-pass filter, a bandpass filter, and a high-pass filter. This approach would give greater control over the ripple amplitudes and the transition widths but would require a trial and error approach to establish the specifications for the three filters.

9.12 Shown in the following figure is the magnitude of the frequency response of a type I high-pass filter that was designed using the Parks-McClellan algorithm.
The stopband cutoff frequency is $\omega_s = 0.4\pi$, and the passband cutoff frequency is $\omega_p = 0.5\pi$. In addition, the stopband ripple is $\delta_s = 0.0574$, and the passband ripple is $\delta_p = 0.1722$.

(a) Determine the weighting function, $W(e^{j\omega})$, used to design this filter, and find the length of the unit sample response.

(b) Describe approximately where the zeros of the system function of this filter lie in the $z$-plane.

(a) To determine the weighting function, we observe that

$$\frac{\delta_p}{\delta_s} = 3$$

Therefore, the weighting function used to design the filter has a value in the stopband that is 3 times larger than the value in the passband. This makes the errors in the stopband more costly and, therefore, smaller by a factor of 3. So, a weighting function that could have been used to design this filter is as follows:

$$W(e^{j\omega}) = \begin{cases} 
3 & 0 \leq \omega \leq 0.4\pi \\
1 & 0.5\pi \leq \omega \leq \pi 
\end{cases}$$

To determine the length of the unit sample response, recall that a type I equiripple high-pass (or low-pass) filter must either have $L + 2$ or $L + 3$ alternations where $L = N/2$. Therefore, the order of the filter may be determined by counting the alternations. For this filter, we have nine alternations, which are labeled in the figure below.
Thus, \( L = 7 \) or, in the case of an extraripple filter, \( L = 6 \). However, in order for \( h(n) \) to be an extraripple filter, \( \omega = 0 \) and \( \omega = \pi \) must both be extremal frequencies (see Prob. 9.15). Because \( \omega = 0 \) is not an extremal frequency, this is not an extraripple filter. Therefore, \( L = 7 \) and \( N = 14 \).

(b) Because the order of this filter is \( N = 14 \), the system function has 14 zeros. For a linear phase filter, we know that the zeros of the system function may lie on the unit circle, or they may occur in conjugate reciprocal pairs. From the plot of the frequency response magnitude, we see that \(|H(e^{j\omega})| = 0 \) at \( \omega_1 \approx 0.175\pi \), \( \omega_2 \approx 0.3\pi \), and \( \omega_3 \approx 0.39\pi \). Therefore, there are three zeros on the unit circle at these frequencies. Because there must also be zeros at the conjugate positions, \( z = e^{-j\omega_i} \) for \( i = 1, 2, 3 \), these unit circle zeros account for six of the fourteen zeros. In addition to these, there must be a conjugate pair of zeros at \( z = re^{j\omega_4} \), where \( \omega_4 \approx 0.7\pi \). These zeros account for the dip in \(|H(e^{j\omega})|\) at \( \omega = 0.7\pi \). Because the filter has linear phase, in addition to this pair of complex zeros, there must be a pair at the reciprocal locations, \( z^{-1} = re^{-j\omega_4} \). For the same reason, there will be zeros on the real axis at \( z = \alpha_1 \) and \( z = 1/\alpha_1 \), as well as zeros on the real axis at \( z = -\alpha_2 \) and \( z = -1/\alpha_2 \), where \( \alpha_1 \) and \( \alpha_2 \) are positive real numbers that are less than 1. These four zeros account for the minima in \(|H(e^{j\omega})|\) at \( \omega = 0 \) and \( \omega = \pi \). A plot showing the actual positions of the 14 zeros of \( H(z) \) is given below.

9.13 With the frequency sampling method, the frequency samples match the ideal frequency response exactly.

Derive an interpolation formula that shows how the frequency samples \( H(k) \) are interpolated.

The frequency response of an FIR filter of length \( N \) is

\[
H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}
\]

If \( h(n) \) is designed using the frequency sampling method,

\[
H(k) = \sum_{n=0}^{N-1} h(n)e^{-j2\pi nk/N} = H_d(e^{j2\pi k/N}) \quad k = 0, 1, \ldots, N - 1
\]

Because these frequency samples correspond to the \( N \)-point DFT of \( h(n) \), the unit sample response may be expressed in terms of these samples as follows:

\[
h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j2\pi nk/N}
\]

Substituting this into the expression above for \( H(e^{j\omega}) \), we have

\[
H(e^{j\omega}) = \sum_{k=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j2\pi nk/N} \right\} e^{-j\omega N} = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \left\{ \sum_{n=0}^{N-1} e^{-j\omega(2\pi n/N)} \right\}
\]

Using the geometric series to evaluate this sum, we find

\[
H(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{1 - e^{-j(\omega-2\pi k/N)N}}{1 - e^{-j(\omega-2\pi k/N)}} = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{\sin \frac{N}{2} \left( \frac{\omega - 2\pi k}{N} \right)}{\sin \frac{1}{2} \left( \frac{\omega - 2\pi k}{N} \right)} e^{-j\frac{2\pi k}{N}(\omega-2\pi k)}
\]

which is the desired interpolation formula.
9.14 Given a low-pass filter that has been designed and implemented, either in hardware or software, it may be of interest to try to improve the frequency response characteristics by repetitive use of the filter. Suppose that \( h(n) \) is the unit sample response of a zero phase FIR filter with a frequency response that satisfies the following specifications:

\[
1 - \delta_p < H(e^{j\omega}) < 1 + \delta_p \quad 0 \leq \omega \leq \omega_p \\
-\delta_s < H(e^{j\omega}) < \delta_s \quad \omega_s \leq \omega \leq \pi
\]

(Note that \( H(e^{j\omega}) \) having zero phase implies that \( H(e^{j\omega}) \) is real-valued for all \( \omega \)).

(a) If a new filter is formed by cascading \( h(n) \) with itself,

\[
g(n) = h(n) * h(n)
\]

\( G(e^{j\omega}) \) satisfies a set of specifications of the form

\[
A < G(e^{j\omega}) < B \quad 0 \leq \omega \leq \omega_p \\
C < G(e^{j\omega}) < D \quad \omega_s \leq \omega \leq \pi
\]

Find \( A, B, C, \) and \( D \) in terms of \( \delta_p \) and \( \delta_s \) of the low-pass filter \( h(n) \).

(b) If \( \delta_p \ll 1 \) and \( \delta_s \ll 1 \), what are the approximate passband and stopband ripples of \( G(e^{j\omega}) \)?

(a) With a cascade, \( g(n) = h(n) * h(n) \), the frequency response is

\[
G(e^{j\omega}) = H^2(e^{j\omega})
\]

Therefore, in the passband we have

\[
(1 - \delta_p)^2 < G(e^{j\omega}) < (1 + \delta_p)^2 \quad 0 \leq \omega \leq \omega_p
\]

and in the stopband we have

\[
0 \leq G(e^{j\omega}) < \delta_s^2 \quad \omega_s \leq \omega \leq \pi
\]

(b) If we assume that \( \delta_p \ll 1 \),

\[
(1 - \delta_p)^2 = 1 - 2\delta_p + \delta_p^2 \approx 1 - 2\delta_p
\]

and

\[
(1 + \delta_p)^2 = 1 + 2\delta_p + \delta_p^2 \approx 1 + 2\delta_p
\]

Therefore, the passband specifications are approximately

\[
1 - 2\delta_p < G(e^{j\omega}) < 1 + 2\delta_p \quad 0 \leq \omega \leq \omega_p
\]

In other words, the passband ripple is approximately doubled. In the stopband, however, the ripple is much smaller with the cascade. In fact, in decibels, the stopband attenuation is doubled. With other systems built out of interconnections of \( h(n) \) it is possible to improve the filter characteristics in both the passband and the stopband.

9.15 Show that a type I equiripple low-pass filter of order \( N \) may have either \( L + 2 \) or \( L + 3 \) alternations where \( L = N/2 \).

For a type I linear phase filter of order \( N \), the frequency response is

\[
H(e^{j\omega}) = A(e^{j\omega})e^{-jN\omega/2}
\]

where

\[
A(e^{j\omega}) = \sum_{k=-L}^{L} a(k) \cos k\omega
\]
FILTER DESIGN

with $L = N/2$. Because the desired response, $A_0(e^{j\omega})$, and the weighting function, $W(e^{j\omega})$, are piecewise constant,

$$\frac{dE(e^{j\omega})}{d\omega} = \frac{d}{d\omega} [W(e^{j\omega})[A_0(e^{j\omega}) - A(e^{j\omega})]] = -\frac{dA(e^{j\omega})}{d\omega}$$

However, because $A(e^{j\omega})$ is a trigonometric polynomial of degree $N/2$ in $\cos \omega$,

$$A(e^{j\omega}) = \sum_{k=0}^{L-1} a(k) \cos k\omega = \sum_{k=0}^{L-1} a(k)(\cos \omega)^k$$

then

$$\frac{dA(e^{j\omega})}{d\omega} = -\sin \omega \sum_{k=0}^{L-1} ka(k)(\cos \omega)^{k-1} = -\sin \omega \sum_{k=0}^{L-1} (k+1)a(k+1)(\cos \omega)^k$$

Therefore, the derivative of $A(e^{j\omega})$ with respect to $\omega$ is always equal to zero at $\omega = 0$ and $\omega = \pi$. In addition, however, the derivative is equal to zero at $L-1$ other frequencies between 0 and $\pi$, which correspond to the roots of the polynomial given by the sum. Therefore, $A(e^{j\omega})$ may have at most $L + 1$ local maxima and minima. In addition, however, the band-edge frequencies, $\omega_c$, and $\omega_s$, must also be extremal frequencies. Thus, the maximum number of alternations is $L + 3$. Because the alternation theorem requires a minimum of $L + 2$, the optimum filter may have either $L + 2$ or $L + 3$ alternations.

9.16 Suppose that we would like to design a type I equiripple bandstop filter of order $N = 30$.

(a) What is the minimum number of alternations that this filter must have?

(b) What is the maximum number?

(a) For a type I linear phase FIR filter of order $N$, the alternation theorem states that the minimum number of alternations is $L + 2$, where $L = N/2$. Therefore, with $N = 30$, the minimum number of alternations is 17.

(b) As shown in Prob. 9.15, with

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega N/2}$$

$A(e^{j\omega})$, and thus $E(e^{j\omega})$, will have, at most, $L + 1$ local maxima and minima in the interval $[0, \pi]$. In addition, however, there may be alternations at the band-edge frequencies. For a bandstop filter, there are four band edges: the lower passband cutoff frequency, the lower stopband cutoff frequency, the upper stopband cutoff frequency, and the upper passband cutoff frequency. Therefore, the maximum number of alternations is $L + 5$.

9.17 We would like to design a bandstop filter to satisfy the following specifications:

$$0.98 \leq |H(e^{j\omega})| \leq 1.02 \quad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| < 0.001 \quad 0.22\pi \leq \omega \leq 0.78\pi$$

$$0.95 \leq |H(e^{j\omega})| \leq 1.05 \quad 0.8\pi \leq \omega \leq \pi$$

(a) Estimate the order of the equiripple filter required to meet these specifications.

(b) What weighting function $W(e^{j\omega})$ should be used to design this filter?

(c) What is the minimum number of extremal frequencies that the optimal filter must have?

(a) The design formula used to estimate the order for a low-pass equiripple filter is

$$N = \frac{-10 \log(\delta_p \delta_s) - 13}{14.6 \Delta f}$$

With the smaller of the two passband ripples being equal to $\delta_p = 0.02$, a stopband ripple of $\delta_s = 0.001$, and a transition width $\Delta \omega = 0.02\pi$, an estimate of the filter order is

$$N = \frac{-10 \log(0.02 \cdot 0.001) - 13}{14.6 \cdot (0.01)} = 232$$

However, because this estimate is for a low-pass filter, the actual filter order required is closer to $N = 242$, which may be confirmed by computer.
With a ripple of $\delta_1 = 0.02$ in the lower passband, $\delta_2 = 0.001$ in the stopband, and $\delta_3 = 0.05$ in the upper passband, an appropriate weighting function would be

$$W(e^{j\omega}) = \begin{cases} 
1 & 0 \leq \omega \leq 0.2\pi \\
20 & 0.22\pi \leq \omega \leq 0.78\pi \\
0.4 & 0.8\pi \leq \omega \leq \pi 
\end{cases}$$

However, scaling these weights by any constant would not change the design.

Assuming a filter order of $N = 232$, which is a type I design, the amplitude response has the form

$$A(e^{j\omega}) = \sum_{k=0}^L a(k) \cos k\omega$$

where $L = N/2 = 116$. Therefore, the minimum number of extremal frequencies is $L + 2 = 118$.

We would like to design an equiripple high-pass filter of order $N = 64$. The stopband ripple is to be no larger than $\delta_s = 0.001$, and the passband ripple is no larger than $\delta_p = 0.01$. If we want a passband cutoff frequency equal to $\omega_p = 0.72\pi$, what will be the stopband cutoff frequency approximately equal to?

For an equiripple low-pass filter, an approximate relation between the filter order $N$, the passband and stopband ripples, $\delta_p$ and $\delta_s$, respectively, and the transition width $\Delta f$, is given by

$$N = \frac{-10 \log(\delta_p/\delta_s) - 13}{14.6\Delta f}$$

Because a high-pass filter may be formed from a low-pass filter as follows,

$$h_{HP}(n) = \delta \left( n - \frac{N}{2} \right) - h_{LP}(n)$$

this formula is also applicable to high-pass filters. With $N = 64$, $\delta_p = 0.01$, and $\delta_s = 0.001$, we find that

$$\Delta f = \frac{37}{(14.6)(64)} = 0.0396$$

Therefore, if the passband cutoff frequency is $\omega_p = 0.72\pi$, the stopband cutoff frequency will be approximately

$$\omega_s = \omega_p - 2\pi \Delta f = 0.6408\pi$$

Suppose that we want to design a low-pass filter of order $N = 63$ with a cutoff frequency $\omega_p = 0.3\pi$ and a stopband cutoff frequency $\omega_s = 0.32\pi$.

(a) What is the approximate stopband attenuation that would obtained if this filter were designed using the window design method with a Kaiser window.

(b) Repeat part (a) for an equiripple filter assuming that we want $\delta_p = \delta_s$.

(a) For a Kaiser window design, the relationship between the filter order $N$, the stopband attenuation $\alpha_s = -20 \log \delta_s$, and the transition width $\Delta f$ is

$$N = \frac{\alpha_s - 7.95}{14.36\Delta f}$$

Solving this for the stopband attenuation, we have

$$\alpha_s = 14.36 \cdot N \Delta f + 7.95 = 16.99$$

which corresponds to a stopband (and passband) ripple of

$$\delta_s = 10^{-16.99/20} = 0.141$$
(b) For an equiripple filter, the filter order is approximately

\[ N = \frac{-10 \log(\delta_p \delta_s) - 13}{14.6 \Delta f} \]

With \( \delta_p = \delta_s \), this becomes

\[ N = \frac{\alpha_s - 13}{14.6 \Delta f} \]

where \( \alpha_s = -20 \log \delta_s \). Solving for \( \alpha_s \), we have

\[ \alpha_s = 14.6 \cdot N \Delta f + 13 = 22.04 \]

The corresponding stopband ripple is

\[ \delta_s = 10^{-22.04/20} = 0.079 \]

9.20 The linear phase constraint on FIR filters places constraints on the unit sample response and the location of the zeros of the system function. In the table below, indicate with a check which filter types could successfully be used to approximate the given filter type.

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Type I</th>
<th>Type II</th>
<th>Type III</th>
<th>Type IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass filter</td>
<td></td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>High-pass filter</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bandpass filter</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Bandstop filter</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Differentiator</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>

A type I linear phase filter has no constraints on the locations of its zeros. Therefore, a type I filter may be used for the design of any type of filter. The type II linear phase filter will always have a zero at \( \omega = \pi \). Therefore, these filters should only be used for low-pass and bandpass filters. The type III linear phase filter is constrained to have zeros at \( \omega = \pi \) and \( \omega = 0 \). Therefore, type III filters should only be used for the design of bandpass filters. Finally, because the type IV filters have a zero at \( \omega = 0 \), they should not be used in the design of low-pass or bandstop filters. These results are summarized in the table below.

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Type I</th>
<th>Type II</th>
<th>Type III</th>
<th>Type IV</th>
</tr>
</thead>
<tbody>
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<td>x</td>
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</tr>
<tr>
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<tr>
<td>Bandpass filter</td>
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<tr>
<td>Bandstop filter</td>
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<tr>
<td>Differentiator</td>
<td>x</td>
<td></td>
<td>x</td>
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</tr>
</tbody>
</table>

IIR Filter Design

9.21 For historical reasons, the design formulas for analog filters are given assuming a peak gain of 1 in the passband. In terms of the parameters \( \epsilon \) and \( A \), the filter specifications have the form

\[ \frac{1}{\sqrt{1 + \epsilon^2}} \leq |H_a(j\Omega)| \leq 1 \]

\[ |H_a(j\Omega)| \leq \frac{1}{A} \]

Suppose that we would like to use the bilinear transformation to design a discrete-time IIR low-pass filter that satisfies the following frequency response constraints:

\[ 1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p \quad 0 \leq \omega \leq \omega_p \]

\[ |H(e^{j\omega})| \leq \delta_s \quad \omega_s \leq \omega \leq \pi \]
Find the relationship between the parameters $\delta_p$ and $\delta_s$ for the discrete-time filter and between the parameters $\epsilon$ and $A$ for the continuous-time filter.

For a digital low-pass filter with a frequency response magnitude

$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p$$

dividing by $1 + \delta_p$ this becomes

$$\frac{1 - \delta_p}{1 + \delta_p} \leq \frac{|H(e^{j\omega})|}{1} \leq 1$$

Setting

$$\frac{1 - \delta_p}{1 + \delta_p} = \frac{1}{\sqrt{1 + \epsilon^2}}$$

we have

$$1 + \epsilon^2 = \left( \frac{1 + \delta_p}{1 - \delta_p} \right)^2$$

and

$$\epsilon^2 = \left( \frac{1 + \delta_p}{1 - \delta_p} \right)^2 - 1 = \frac{4\delta_p}{(1 - \delta_p)^2}$$

With a stopband ripple of $\delta_s$, the normalization of the peak passband gain to 1 produces a peak stopband ripple of

$$\frac{\delta_s}{1 + \delta_p}$$

Therefore

$$A = \delta_s^{-1}(1 + \delta_p)$$

9.22 As the order of an analog Butterworth filter is increased, the slope of $|H_a(j\Omega)|^2$ at the 3-dB cutoff frequency, $\Omega_c$, increases. Derive an expression for the slope of $|H_a(j\Omega)|^2$ at $\Omega_c$ as a function of the filter order, $N$.

The magnitude squared of the Butterworth filter’s frequency response is

$$|H_a(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}$$

To evaluate the slope of $|H_a(j\Omega)|^2$ at $\Omega = \Omega_c$, we may set $\Omega_c = 1$ and evaluate the derivative at $\Omega = 1$. Therefore, with

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$$

we have

$$\frac{d}{d\Omega} |H_a(j\Omega)|^2 = -\frac{2N\Omega^{2N-1}}{(1 + \Omega^{2N})^2}$$

Evaluating this at $\Omega = 1$, we have

$$\frac{d}{d\Omega} |H_a(j\Omega)|^2 \bigg|_{\Omega=1} = -\frac{N}{2}$$

9.23 Show that the frequency response of an $N$th-order low-pass Butterworth filter is maximally flat at $\Omega = 0$ in the sense that the first $2N - 1$ derivatives of $|H_a(j\Omega)|^2$ are equal to zero at $\Omega = 0$.

An $N$th-order Butterworth filter has a magnitude squared frequency response given by

$$|H_a(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}$$
Without any loss in generality, we may assume that $\Omega_c = 1$ and evaluate the derivative of the function

$$G(\Omega) = \frac{1}{1 + \Omega^{2N}}$$

at $\Omega = 0$. Multiplying both sides of this equation by $(1 + \Omega^{2N})$, we have

$$G(\Omega)(1 + \Omega^{2N}) = 1$$

Differentiating this equation with respect to $\Omega$ yields

$$G'(\Omega)(1 + \Omega^{2N}) + G(\Omega)(2N\Omega^{2N-1}) = 0$$

Setting $\Omega = 0$, we have

$$G'(\Omega)|_{\Omega=0} = 0$$

Differentiating a second time gives

$$G''(\Omega)(1 + \Omega^{2N}) + G'(\Omega)(4N\Omega^{2N-1}) + G(\Omega)(2N(2N-1)\Omega^{2N-2}) = 0$$

Again setting $\Omega = 0$, we have

$$G''(\Omega)|_{\Omega=0} = 0$$

If we continue to differentiate $k$ times, where $k \leq 2N - 1$, we have an equation of the form

$$G^{(i)}(\Omega)(1 + \Omega^{2N}) + \sum_{i=1}^{k-1} G^{(i)}(\Omega)F_i(\Omega) + G(\Omega)(2N(2N-1)\cdots(2N-k+1)\Omega^{2N-k}) = 0$$

where $G^{(i)}(\Omega)$ is the $i$th derivative of $G(\Omega)$, and $F(\Omega)$ is a polynomial in $\Omega$. Given that $G^{(i)}(\Omega)$ is equal to zero at $\Omega = 0$ for $i = 1, \ldots, k - 1$, it follows that

$$G^{(k)}(\Omega)|_{\Omega=0} = 0$$

Differentiating $2N$ times, however, we have

$$G^{(2N)}(\Omega)(1 + \Omega^{2N}) + \sum_{i=1}^{2N-1} G^{(i)}(\Omega)F_i(\Omega) + G(\Omega) \cdot (2N)! = 0$$

Therefore,

$$G^{(2N)}(\Omega)|_{\Omega=0} = -G(\Omega)|_{\Omega=0} \cdot (2N)! = -(2N)!$$

which is nonzero, and the maximally flat property is established.

### 9.24 Design a low-pass Butterworth filter that has a 3-dB cutoff frequency of 1.5 kHz and an attenuation of 40 dB at 3.0 kHz.

Given the 3-dB cutoff frequency of the Butterworth filter, all that is needed is to find the filter order, $N$, that will give 40 dB of attenuation at 3 kHz, or $\Omega_c = 2\pi \cdot 3000$. At the stopband cutoff frequency $\Omega_s$, the magnitude of the frequency response squared is

$$\left|H_o(j\Omega)|_{\Omega=2\pi \cdot 3000}^2 = \left|\frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}\right|_{\Omega=2\pi \cdot 3000}^2 = \frac{1}{1 + 2^{2N}}$$

Therefore, if we want the magnitude of the frequency response to be down 40 dB at $\Omega_s = 2\pi \cdot 3000$, the magnitude squared must be no larger than $10^{-4}$, or

$$\frac{1}{1 + 2^{2N}} \leq 10^{-4}$$

Thus, we want

$$2N = \frac{\log(10^4 - 1)}{\log 2} = 13.29$$
or \( N = 7 \). For a seventh-order Butterworth filter, the 14 poles of

\[
H_a(s)H_a(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}
\]

lie on a circle of radius \( \Omega_c = 2\pi \cdot 3000 \), at angles of

\[
\theta_k = \frac{(N + 1 + 2k)\pi}{N} = \frac{(4 + k)\pi}{7} \quad k = 0, 1, \ldots, 13
\]

as illustrated in the following figure:

The poles of \( H_a(s) \) are the seven poles of \( H_a(s)H_a(-s) \) that lie in the left-half \( s \)-plane, that is,

\[
s_k = -\Omega_c e^{j\theta_k} \quad k = 0, 1, 2, 3
\]

Except for the isolated pole at \( s = -\Omega_c \), the remaining six poles occur in complex conjugate pairs. The conjugate pairs may be combined to form second-order factors with real coefficients to yield factors of the form

\[
H_k(s) = \frac{1}{s^2 - 2\Omega_c \cos(k\pi/7)s + \Omega_c^2} \quad k = 1, 2, 3
\]

Thus, the system function of the seventh-order Butterworth filter is

\[
H_a(s) = \prod_{k=0}^{N-1} \frac{s - s_k}{s - s_k} = \frac{\Omega_c}{s + \Omega_c} \cdot \prod_{k=1}^{3} \frac{s^2 - 2\Omega_c \cos(k\pi/7)s + \Omega_c^2}{s^2 - \Omega_c^2}
\]

9.25 Let \( \Omega_p \) and \( \Omega_s \) be the desired passband and stopband cutoff frequencies of an analog low-pass filter, and let \( \delta_p \) and \( \delta_s \) be the passband and stopband ripples. Show that the order of the Butterworth filter required to meet these specifications is

\[
N \geq \frac{\log d}{\log k}
\]

with the 3-dB cutoff frequency \( \Omega_c \) being any value within the range

\[
\Omega_p[(1 - \delta_p)^{-2} - 1]^{-1/2N} \leq \Omega_c \leq \Omega_s[\delta_s^{-2} - 1]^{-1/2N}
\]

The squared magnitude of the frequency response of the Butterworth filter is

\[
|H_a(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}
\]
Because \( |H_a(j\Omega)| \) is monotonically decreasing, the maximum error in the passband and stopband occurs at the band edges, \( \Omega_p \) and \( \Omega_s \), respectively. Therefore, we want

\[
|H_a(j\Omega_p)|^2 = \frac{1}{1 + (j\Omega_p/j\Omega_c)^{2N}} \geq (1 - \delta_p)^2
\]

and

\[
|H_a(j\Omega_s)|^2 = \frac{1}{1 + (j\Omega_s/j\Omega_c)^{2N}} \leq \delta_s^2
\]

From the first equation, we have

\[
\left( \frac{\Omega_p}{\Omega_c} \right)^{2N} \leq (1 - \delta_p)^{-2} - 1
\]

and from the second,

\[
\left( \frac{\Omega_s}{\Omega_c} \right)^{2N} \geq \delta_s^{-2} - 1
\]

Dividing these two equations, we have

\[
\left( \frac{\Omega_p}{\Omega_s} \right)^{2N} \leq \frac{(1 - \delta_p)^{-2} - 1}{\delta_s^{-2} - 1} = d^2
\]

and taking the logarithm gives

\[
N \log \left( \frac{\Omega_p}{\Omega_s} \right) \leq \log d
\]

Dividing by

\[
\log \left( \frac{\Omega_p}{\Omega_s} \right) = \log k
\]

we have

\[
N \geq \frac{\log d}{\log k}
\]

(note that the inequality is reversed because \( \log k < 0 \)). Because the right side of this equation will not generally be an integer, the order \( N \) is taken to be the smallest integer larger than \( \log d/\log k \). Finally, once the order \( N \) is fixed, it follows from Eqs. (9.20) and (9.21) that \( \Omega_c \) may be any value in the range

\[
\Omega_p[(1 - \delta_p)^{-2} - 1]^{-1/2N} \leq \Omega_c \leq \Omega_s\left[\delta_s^{-2} - 1\right]^{-1/2N}
\]

9.26 Suppose that we would like to design an analog Chebyshev low-pass filter so that

\[
1 - \delta_p \leq |H_a(j\Omega)| \leq 1 \quad |\Omega| \leq \Omega_p
\]

\[
|H_a(j\Omega)| \leq \delta_s \quad \Omega_s \leq |\Omega|
\]

Find an expression for the required filter order, \( N \), as a function of \( \Omega_p, \Omega_s, \delta_p, \) and \( \delta_s \).

For a Chebyshev filter, the magnitude of the frequency response squared is

\[
|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_p)}
\]

where

\[
T_N(x) = \begin{cases} 
\cos(N \cos^{-1} x) & |x| \leq 1 \\
\cosh(N \cosh^{-1} x) & |x| > 1 
\end{cases}
\]

is an \( N \)th-order Chebyshev polynomial. Over the passband, \( |\Omega| < \Omega_p \), the magnitude of the frequency response oscillates between 1 and \((1 + \epsilon^2)^{-1/2}\). Therefore, the ripple amplitude, \( \delta_p \), is

\[
\delta_p = 1 - (1 + \epsilon^2)^{-1/2}
\]
or 

\[ \epsilon^2 = (1 - \delta_p)^2 - 1 \]

At the stopband frequency \( \Omega_s \), we have

\[ |H_a(j\Omega_s)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega_s/\Omega_p)} \]

which we want to be less than or equal to \( \delta_s^2 \):

\[ \frac{1}{1 + \epsilon^2 T_N^2(\Omega_s/\Omega_p)} \leq \delta_s^2 \]

Therefore,

\[ T_N^2 \left( \frac{\Omega_s}{\Omega_p} \right) \geq \frac{(\delta_s^{-2} - 1)}{\epsilon^2} = \frac{(\delta_s^{-2} - 1)}{(1 - \delta_p)^2 - 1} = \frac{1}{d^2} \]

Because \((\Omega_s/\Omega_p) > 1\), then \( T_N(\Omega_s/\Omega_p) = \cosh(N \cosh^{-1}(\Omega_s/\Omega_p)) \), and we have

\[ \cosh \left( N \cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right) \right) \geq \frac{1}{d} \]

or

\[ N \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} \]

which is the desired expression.

9.27 If \( H_a(s) \) is a third-order type I Chebyshev low-pass filter with a cutoff frequency \( \Omega_p = 1 \) and \( \epsilon = 0.1 \), find \( H_a(s)H_a(-s) \).

The magnitude of the frequency response squared for an \( N \)th-order type I Chebyshev filter is

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_p)} \]

where \( T_N(x) \) is an \( N \)th-order Chebyshev polynomial that is defined recursively as follows,

\[ T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \quad k \geq 1 \]

with \( T_0(x) = 1 \) and \( T_1(x) = x \). Therefore, to find the third-order Chebyshev polynomial, we first find \( T_2(x) \) as follows,

\[ T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1 \]

and then we have

\[ T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 2x - x = x(4x^2 - 3) \]

Thus, the denominator polynomial in \( |H_a(j\Omega)|^2 \) is

\[ 1 + \epsilon^2 T_3^2 \left( \frac{\Omega}{\Omega_p} \right) = 1 + 0.01[\Omega(4\Omega^2 - 3)]^2 = 1 + 0.01\Omega^2(16\Omega^4 - 24\Omega^2 + 9) \]

and we have

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + 0.09\Omega^2 - 0.24\Omega^4 + 0.16\Omega^6} \]

Because

\[ |H_a(j\Omega)|^2 = |H_a(s)H_a(-s)|_{\Omega \to \Omega} \]

to find the rational function

\[ G_a(s) = H_a(s)H_a(-s) \]

we make the substitution \( \Omega = s/j \) in \( |H_a(j\Omega)|^2 \) as follows:

\[ G_a(s) = \frac{1}{1 + 0.09(s/j)^2 - 0.24(s/j)^4 + 0.16(s/j)^6} = \frac{1}{1 - 0.09s^2 - 0.24s^4 - 0.16s^6} \]
9.28 Show that the bilinear transformation maps the \( j\Omega \)-axis in the \( s \)-plane onto the unit circle, \(|z| = 1\), and maps the left-half \( s \)-plane, \( \text{Re}(s) < 0 \) inside the unit circle, \(|z| < 1\).

To investigate the characteristics of the bilinear transformation, let \( z = re^{j\omega} \) and \( s = \sigma + j\Omega \). The bilinear transformation may then be written as

\[
s = \frac{2}{T_i} \frac{z - 1}{z + 1} = \frac{2}{T_i} \frac{re^{j\omega} - 1}{re^{j\omega} + 1}
\]

\[
= \frac{2}{T_i} \left( \frac{r^2 - 1}{1 + r^2 + 2r \cos \omega} + j \frac{2r \sin \omega}{1 + r^2 + 2r \cos \omega} \right)
\]

Therefore,

\[
\sigma = \frac{2}{T_i} \frac{r^2 - 1}{1 + r^2 + 2r \cos \omega}
\]

and

\[
\Omega = \frac{2}{T_i} \frac{2 \sin \omega}{1 + r^2 + 2r \cos \omega}
\]

Note that if \( r < 1 \), then \( \sigma < 0 \), and if \( r > 1 \), then \( \sigma > 0 \). Consequently, the left-half \( s \)-plane is mapped inside the unit circle, and the right-half \( s \)-plane is mapped outside the unit circle. If \( r = 1 \), then \( \sigma = 0 \), and

\[
\Omega = \frac{2}{T_i} \frac{2 \sin \omega}{1 + 2 \cos \omega}
\]

Thus, the \( j\Omega \)-axis is mapped onto the unit circle. Using trigonometric identities, this may be written in the equivalent form

\[
\Omega = \frac{2}{T_i} \tan \left( \frac{\omega}{2} \right)
\]

or

\[
\omega = 2 \arctan \left( \frac{\Omega T_i}{2} \right)
\]

9.29 Let \( H_p(s) \) be an all-pole filter with no zeros in the finite \( s \)-plane,

\[
H_p(s) = A \prod_{k=1}^{n} \frac{1}{s - s_k}
\]

If \( H_p(s) \) is mapped into a digital filter using the bilinear transformation, will \( H(z) \) be an all-pole filter?

With \( T_i = 2 \), the bilinear transformation is

\[
s = \frac{1 - z^{-1}}{1 + z^{-1}}
\]

and the system function of the digital filter is

\[
H(z) = A \prod_{k=1}^{n} \frac{1}{1 - \frac{z^{-1}}{1 - z^{-1}} - s_k} = A \prod_{k=1}^{n} \frac{(1 + z^{-1})}{1 - z^{-1} - s_k (1 + z^{-1})}
\]

\[
= A \prod_{k=1}^{n} \frac{1 + z^{-1}}{(1 - s_k) - (1 + s_k)z^{-1}}
\]

This may be written in the more conventional form as follows,

\[
H(z) = A' \prod_{k=1}^{n} \frac{1 + z^{-1}}{1 - s_k z^{-1}}
\]

where

\[
a' = \prod_{k=1}^{n} \frac{1}{1 - s_k}
\]
and

\[ a_k = \frac{1 + s_k}{1 - s_k} \]

Therefore, \( H(z) \) has \( p \) poles (inside the unit circle if \( \text{Re}(s_k) < 0 \)) and \( p \) zeros at \( z = -1 \). Note that these zeros come from the \( p \) zeros in \( H_a(s) \) at \( s = \infty \), which are mapped to \( z = -1 \) by the bilinear transformation. Thus, \( H(z) \) will not be an all-pole filter.

9.30 Shown in the figure below is the magnitude of the frequency response of a low-pass filter that was designed by mapping a type I analog Chebyshev filter into a discrete-time filter using the bilinear transformation.

Find the filter order (i.e., the number of poles and zeros in \( H(z) \)).

The magnitude-squared response of a type I analog Chebyshev filter is

\[ |H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_p)} \]

where

\[ T_N(x) = \begin{cases} \cos(N \cos^{-1} x) & |x| \leq 1 \\ \cosh(N \cosh^{-1} x) & |x| > 1 \end{cases} \]

is an \( N \)th-order Chebyshev polynomial. Over the passband, \( |\Omega| < \Omega_p \), the magnitude of the frequency response oscillates between 1 and \((1 + \epsilon^2)^{-1/2}\). As the frequency varies from \( \Omega = 0 \) to \( \Omega = \Omega_p \), \( \theta = N \cos^{-1}(\Omega/\Omega_p) \) varies from \( \theta = N\pi/2 \) to \( \theta = 0 \). Therefore,

\[ T_N^2 \left( \frac{\Omega}{\Omega_p} \right) = \cos \left[ N \cos^{-1} \left( \frac{\Omega}{\Omega_p} \right) \right] \]

oscillates between zero and \( 1 \) \( N + 1 \) times over the interval \([0, \Omega_p]\) [i.e., \( T_N^2(\Omega/\Omega_p) \) reaches its maximum or minimum value \( N + 1 \) times]. The bilinear transformation is a one-to-one mapping of the \( j\Omega \)-axis onto the unit circle. Therefore, \( |H(e^{j\omega})|^2 \) will alternate \( N + 1 \) times between 1 and \( 1/(1 + \epsilon^2) \) over the interval \([0, \omega_p]\), where

\[ \omega_p = \tan \left( \frac{\Omega_p}{2} \right) \]

Because there are six alternations of \( |H(e^{j\omega})|^2 \) in the passband, \( N + 1 = 6 \), and \( H_a(s) \) is a fifth-order type I Chebyshev filter,

\[ H_a(s) = A \prod_{k=1}^{5} \frac{1}{s - s_k} \]

where \( A \) is a constant. Applying the bilinear transformation to \( H_a(s) \) results in a discrete-time filter with a system function \( H(z) \) that has five poles inside the unit circle, and five zeros on the unit circle at \( z = -1 \) (as shown in Prob. 9.29, the five zeros come from the five zeros in \( H_a(s) \) at \( s = \infty \)).
9.31 Use the bilinear transformation to design a discrete-time Chebyshev high-pass filter with an equiripple passband with

\[ 0 \leq |H(e^{j\omega})| \leq 0.1 \quad 0 \leq \omega \leq 0.1\pi \]

and

\[ 0.9 \leq |H(e^{j\omega})| \leq 1.0 \quad 0.3\pi \leq \omega \leq \pi \]

To design a discrete-time high-pass filter, there are two approaches that we may use. We may design an analog type I Chebyshev low-pass filter, map it into a Chebyshev low-pass filter using the bilinear transformation, and then perform a low-pass-to-high-pass transformation in the \( z \)-domain. Alternatively, before applying the bilinear transformation, we could perform a low-pass-to-high-pass transformation in the \( s \)-plane and then map the analog high-pass filter into a discrete-time high-pass filter using the bilinear transformation. Because both methods result in the same design, it does not matter which method we use. Therefore, we will use the second approach, because it is a little easier algebraically.

Given that we want to design a high-pass filter with a stopband cutoff frequency \( \omega_s = 0.117 \) and a passband cutoff frequency \( \omega_p = 0.317 \), we first transform the specifications of the digital filter into the continuous-time domain. With \( T_s = 2 \) and

\[ \Omega = \tan \frac{\omega}{2} \]

we have

\[ \Omega_s = \tan \frac{\omega_s}{2} = \tan(0.05\pi) = 0.1584 \]

and

\[ \Omega_p = \tan \frac{\omega_p}{2} = \tan(0.15\pi) = 0.5095 \]

Using the transformation \( s \rightarrow \frac{1}{s} \) to map these high-pass filter cutoff frequencies to low-pass filter cutoff frequencies, we have

\[ \Omega_p = \frac{1}{0.5095} = 1.9627 \]

and

\[ \Omega_s = \frac{1}{0.1584} = 6.3138 \]

Therefore, the selectivity factor for the analog Chebyshev filter is

\[ k = \frac{\Omega_p}{\Omega_s} = 0.3109 \]

With \( \delta_p = \delta_s = 0.1 \), the discrimination factor is

\[ d = \left[ \frac{(1 - \delta_p)^2 - 1}{\delta_s^2 - 1} \right]^{1/2} = 0.04867 \]

Thus, the required filter order is

\[ N = \cosh^{-1}(1/d) = 2.03 \]

Although we should round up to \( N = 3 \), with a second-order Chebyshev filter we should come close to meeting the specifications. Therefore, we will use \( N = 2 \).

The next step is to design a second-order low-pass Chebyshev filter with

\[ 0.9 \leq |H_0(j\Omega)| \leq 1 \quad 0 \leq \Omega \leq \Omega_p \]

\[ |H_0(j\Omega)| \leq 0.1 \quad \Omega_s \leq \Omega \]

where \( \Omega_p = 1.9627 \) and \( \Omega_s = 6.3138 \). With

\[ \frac{1}{\sqrt{1 + \epsilon^2}} = 1 - \delta_p \]
it follows that
\[ \epsilon^2 = \frac{1}{(1-\delta_p)^2} - 1 = 0.2346 \]

For a second-order Chebyshev filter, we need to generate a second-order Chebyshev polynomial, which is
\[ T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1 \]

Squaring \( T_2(x) \), we have
\[ T_2^2(x) = 4x^4 - 4x^2 + 1 \]

and, for the magnitude squared frequency response of the Chebyshev filter, we have
\[
|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_2^2(\Omega/\Omega_p)} = \frac{1}{1 + \epsilon^2(4(\Omega/\Omega_p)^2 - 4(\Omega/\Omega_p)^2 + 1)}
\]

Substituting for the given values of \( \Omega_p \) and \( \epsilon \), we have
\[
|H_a(j\Omega)|^2 = \frac{1}{1.2346 - 0.2436\Omega^2 + 0.0632\Omega^4}
\]

Next, we find \( H_a(s)H_a(-s) \) with the substitution \( \Omega = -js \),
\[
H_a(s)H_a(-s) = |H_a(j\Omega)|^2_{\Omega=-js} = \frac{1}{1.2346 + 0.2436 s^2 + 0.0632 s^4}
\]

Factoring the denominator polynomial, we find that the two roots in the left-half \( s \)-plane are at
\[
s_1 = -1.1163 + j1.7811 \quad s_2 = s_1^* = -1.1163 - j1.7811
\]

Thus, the second-order Chebyshev filter is
\[
H_a(s) = \frac{1}{\sqrt{1 - \epsilon^2}} \frac{s_1s_1^*}{(s - s_1)(s - s_1^*)} = \frac{3.9778}{s^2 + 2.2327s + 4.4185}
\]

Now we transform this low-pass filter into a high-pass filter with the low-pass-to-high-pass transformation \( s \rightarrow 1/s \). This gives
\[
H_a(s) = \frac{3.9778s^2}{1 + 2.2327s + 4.4185s^2}
\]

Finally, applying the bilinear transformation, we have
\[
H(z) = \frac{3.9778\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2}{1 + 2.2327\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 4.4185\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2}
\]

Multiplying numerator and denominator by \( (1 + z^{-1})^3 \) gives
\[
H(z) = \frac{3.9778(1 - z^{-1})^2}{(1 + z^{-1})^2 + 2.2327(1 + z^{-1})(1 - z^{-1}) + 4.4185(1 - z^{-1})^2}
\]
\[
= \frac{0.52(1 - 2z^{-1} + z^{-2})}{1 - 0.894z^{-1} + 0.4164z^{-2}}
\]

The magnitude of the frequency response is plotted in the following figure.
As a check on the design, we may compute the magnitude of the frequency response at \( \omega = 0.1\pi \), which is

\[ |H(e^{j\omega})|_{\omega=0.1\pi} = 0.1044 \]

which comes close to satisfying the stopband specifications. At the passband edge, we have

\[ |H(e^{j\omega})|_{\omega=0.3\pi} = 0.9044 \]

which does satisfy the constraint.

9.32 We would like to design a digital low-pass filter that has a passband cutoff frequency \( \omega_p = 0.375\pi \) with \( \delta_p = 0.01 \) and a stopband cutoff frequency \( \omega_s = 0.5\pi \) with \( \delta_s = 0.01 \). The filter is to be designed using the bilinear transformation. What order Butterworth, Chebyshev, and elliptic filters are necessary to meet the design specifications?

To find the required filter order, we begin by finding the discrimination factor and the selectivity factor for the analog low-pass filter prototype. With \( \delta_p = \delta_s = 0.01 \), the discrimination factor is

\[ d = \left[ \frac{(1 - \delta_p)^{-2}}{\delta_s^{-2}} - 1 \right]^{1/2} = \left[ \frac{(0.99)^{-2} - 1}{(0.01)^{-2} - 1} \right]^{1/3} = 1.425 \times 10^{-3} \]

For the selectivity factor, we first find the cutoff frequencies for the analog prototype. With \( \omega_p = 0.375\pi \) and \( \omega_s = 0.5\pi \), we prewarp the frequencies as follows (\( T = 2 \)),

\[ \Omega_p = \tan \left( \frac{0.375\pi}{2} \right) = 0.6682 \]
\[ \Omega_s = \tan \left( \frac{0.5\pi}{2} \right) = 1 \]

Therefore,

\[ k = \frac{\Omega_p}{\Omega_s} = 0.6682 \]

For the Butterworth filter, the required filter order is

\[ N = \frac{\log d}{\log k} = 16.25 \]

or \( N = 17 \). For the Chebyshev filter,

\[ N = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = 7.55 \]

so the minimum order is \( N = 8 \). Finally, for the elliptic filter, we first evaluate

\[ q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} \]

where

\[ q_0 = \frac{1}{2} \frac{1 - (1 - k^2)^{1/4}}{1 + (1 - k^2)^{1/4}} \]
With \( k = 0.6682 \), we have

\[
q_0 = \frac{1}{2} \left( \frac{1 - (1 - 0.6682^2)^{1/4}}{1 + (1 - 0.6682^2)^{1/4}} \right) = 0.0369
\]

and

\[
q = 0.0369
\]

Therefore, for the filter order, we find

\[
N = \frac{\log(16/d^2)}{\log(1/q)} = 4.81
\]

or \( N = 5 \).

**9.33** With impulse invariance, a first-order pole in \( H_a(s) \) at \( s = s_k \) is mapped to a pole in \( H(z) \) at \( z = e^{s_k T_s} \):

\[
\frac{1}{s - s_k} \implies \frac{1}{1 - e^{s_k T_s} z^{-1}}
\]

Determine how a second-order pole is mapped with impulse invariance.

If the system function of a continuous-time filter is

\[ H_a(s) = \frac{1}{(s - s_k)^2} \]

the impulse response is

\[ h_a(t) = t e^{s_k t} u(t) \]

where \( u(t) \) is the unit step function. Sampling \( h_a(t) \) with a sampling period \( T_s \), we have

\[ h(n) = h_a(nT_s) = nT_s e^{s_k nT_s} u(n) \]

Using the z-transform property

\[ nX(z) \leftarrow z \frac{dX(z)}{dz} \]

and the z-transform pair

\[ \alpha^n u(n) \leftarrow z \frac{1}{1 - \alpha z^{-1}} \]

it follows that the z-transform of \( h(n) \) is

\[ H(z) = -T_s z \frac{d}{dz} \left[ \frac{1}{1 - e^{s_k T_s} z^{-1}} \right] = \frac{T_s e^{s_k T_s} z^{-1}}{(1 - e^{s_k T_s} z^{-1})^2} \]

Therefore, for a second-order pole, we have the mapping

\[
\frac{1}{(s - s_k)^2} \implies \frac{T_s e^{s_k T_s} z^{-1}}{(1 - e^{s_k T_s} z^{-1})^2}
\]

**9.34** Suppose that we would like to design and implement a low-pass filter with

\[
0.99 \leq |H(e^{j\omega})| \leq 1.01 \quad 0 \leq \omega \leq 0.40\pi
\]

\[
|H(e^{j\omega})| \leq 0.001 \quad 0.42\pi \leq \omega \leq \pi
\]

(a) What order FIR equiripple filter is required to satisfy these specifications?

(b) Repeat part (a) for an elliptic filter.

(c) Compare the complexity of the implementations for the equiripple and elliptic filters in terms of the number of coefficients that must be stored, the number of delays that are required, and the number of multiplications necessary to compute each output sample \( y(n) \).

(a) With a transition width of \( \Delta \omega = 0.02\pi \), an estimate of the required filter order for an FIR equiripple filter is

\[
N = \frac{-10 \log(0.01 \cdot 0.001) - 13}{14.6 \Delta f} = \frac{-10 \log(0.01 \cdot 0.001)}{14.6(0.02\pi)/2\pi} - 13 = \frac{50 - 13}{14.6(0.01)} = 253.4
\]

or \( N = 254 \).
(b) For an elliptic filter, we have
\[ d = \left[ \frac{(1 - \delta_p)^2 - 1}{\delta^2 - 1} \right]^{1/2} = 1.42 \cdot 10^{-4} \]
and
\[ k = \frac{\Omega_p}{\Omega_c} = \frac{\tan(\omega_p/2)}{\tan(\omega_c/2)} = 0.9367 \]
With
\[ q_0 = \frac{1 - (1 - k^2)^{1/4}}{2 + (1 - k^2)^{1/4}} = 0.1282 \]
and
\[ q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} = 0.1283 \]
then
\[ N \geq \frac{\log(16/d^2)}{\log(1/q)} = 9.98 \]
or \( N = 10 \).

(c) For an FIR filter of order \( N = 254 \), the output \( y(n) \) is
\[ y(n) = \sum_{k=0}^{254} h(k)x(n - k) \]
Therefore, implementing this filter requires \( N = 254 \) delays. Since this filter has linear phase, exploiting the symmetry of the unit sample response,
\[ h(n) = h(254 - n) \]
it follows that we must only provide storage for 128 filter coefficients, \( h(0), h(1), \ldots, h(127) \). In addition, we may simplify the evaluation of \( y(n) \) as follows,
\[ y(n) = \sum_{k=0}^{254} h(k)x(n - k) = \sum_{k=0}^{128} h(k)[x(n - k) + x(n - 254 + k)] + h(127)x(n - 127) \]
Thus, 128 multiplications are required to compute each value of \( y(n) \). For a 10th-order elliptic filter,
\[ y(n) = \sum_{k=0}^{10} a(k)y(n - k) + \sum_{k=0}^{10} b(k)x(n - k) \]
Therefore, it follows that 21 memory locations are required to store the coefficients \( a(k) \) and \( b(k) \), and 10 delays are required for a canonic implementation. In addition, we see that 21 multiplications are necessary to evaluate each value of \( y(n) \). However, since the zeros of \( H(z) \) lie on the unit circle, the coefficients \( b(k) \) are symmetric, \( b(k) = b(10 - k) \).

By exploiting this symmetry, we may eliminate five multiplications per output point and five memory locations.

9.35 The input \( x_a(t) \) and output \( y_a(t) \) of a continuous-time filter with a rational system function are related by a linear constant coefficient differential equation of the form
\[ \sum_{k=0}^{p} a(k) \frac{d^k}{dt^k} y_a(t) = \sum_{k=0}^{q} b(k) \frac{d^k}{dt^k} x_a(t) \]
Suppose that we sample \( x_a(t) \) and \( y_a(t) \),
\[ x(n) = x_a(nT_s) \quad y(n) = y_a(nT_s) \]
and approximate a first derivative with the first backward difference,
\[ \frac{d}{dt} x_a(t) \rightarrow \nabla^{(1)} x(n) = \frac{1}{T_s} [x(n) - x(n - 1)] \]
Approximations to higher-order derivatives are then defined as follows,

\[
\frac{d^k}{dt^k} x(t) \rightarrow \nabla^{(k)} x(n) = \nabla[\nabla^{k-1} x(n)]
\]

Applying these approximations to the differential equation gives the following approximation to the differential equation:

\[
\sum_{k=0}^{p} c(k) \nabla^{(k)} y(n) = \sum_{k=0}^{q} d(k) \nabla^{(k)} x(n)
\]

The first backward difference defines a mapping from the \( s \)-plane to the \( z \)-plane that is given by

\[
s = \frac{1 - z^{-1}}{T_s}
\]

Determine the characteristics of this mapping, and compare it to the bilinear transformation. Is this a good mapping to use? Explain why or why not.

As with the bilinear transformation, the first backward difference will map a rational function of \( s \) into a rational function of \( z \). To see how points in the \( s \)-plane map to points in the \( z \)-plane, let us write the mapping as follows,

\[
z = \frac{1}{1 - s T_s}
\]

Note that with \( s = \sigma + j \Omega \),

\[
|z|^2 = \frac{1}{(1 - \sigma T_s)^2 + (\Omega T_s)^2}
\]

and it follows that points in the left-half \( s \)-plane (\( \sigma < 0 \)) are mapped to points inside the unit circle, \( |z| < 1 \). Thus, stable analog filters are mapped to stable digital filters.

Now, let us look at how the \( j\Omega \)-axis is mapped to the \( z \)-plane. With \( s = j \Omega \), we see that

\[
z = \frac{1}{1 - j \Omega T_s} = \frac{1 + j \Omega T_s}{1 + (\Omega T_s)^2}
\]

which is an equation for a circle of radius \( r = \frac{1}{2} \) centered at \( z = \frac{1}{2} \). To see this, note that

\[
z - \frac{1}{2} = \frac{1}{1 - j \Omega T_s} - \frac{1}{2} = \frac{1 + j \Omega T_s}{1 - j \Omega T_s}
\]

Thus,

\[
|z - \frac{1}{2}| = \frac{1}{2} \cdot \frac{|1 + j \Omega T_s|}{1 - j \Omega T_s} = \frac{1}{2}
\]

The properties of the mapping are illustrated in the following figure.

Since the \( j\Omega \)-axis does not map onto the unit circle, the frequency response of the digital filter produced with this mapping will not, in general, be an accurate representation of the frequency response of the analog filter except when \( \omega \) is close to zero. In other words, the frequency response of a continuous-time filter will be well preserved only for low frequencies.
9.36 Use the impulse invariance method to design a digital filter from an analog prototype that has a system function

\[ H_a(s) = \frac{s + a}{(s + a)^2 + b^2} \]

To design a filter using the impulse invariance technique, we first expand \( H_a(s) \) in a partial fraction expansion as follows,

\[
H_a(s) = \frac{s + a}{(s + a)^2 + b^2} = \frac{A_1}{s + (a + jb)} + \frac{A_2}{s + (a - jb)}
\]

where

\[ A_1 = \left( s + a + jb \right) H_a(s) \bigg|_{s=-a+jb} = \frac{s + a}{s + a - jb} \bigg|_{s=-a+jb} = \frac{1}{2} \]

and

\[ A_2 = \left( s + a - jb \right) H_a(s) \bigg|_{s=-a-jb} = \frac{s + a}{s + a + jb} \bigg|_{s=-a-jb} = \frac{1}{2} \]

Therefore, with

\[ H_a(s) = \frac{\frac{1}{2}}{s + a + jb} + \frac{\frac{1}{2}}{s + a - jb} \]

using the mapping given in Eq. (9.10), we have

\[ H(z) = \frac{\frac{1}{2}}{1 - e^{-a-jbT_c}z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-a+jbT_c}z^{-1}} = \frac{1 - e^{-aT_c} \cos(bT_c)z^{-1}}{1 - 2e^{-aT_c} \cos(bT_c)z^{-1} + e^{-2aT_c}z^{-2}} \]

Note that the zero at \( s = -a \) is mapped to a zero at \( z = e^{-aT_c} \cos(bT_c) \). Thus, the location of the zero in the discrete-time filter depends on the position of the poles as well as the zero in the analog filter.

9.37 With the impulse invariance method, the unit sample response of a digital filter is formed by sampling the impulse response of the continuous-time filter,

\[ h(n) = h_a(nT_c) \]

Another approach is to use the step invariance method in which the step response of the digital filter is formed by sampling the step response of the continuous-time filter.

(a) Design a digital filter with the step invariance method using the continuous-time prototype

\[ H_a(s) = \frac{s + a}{(s + a)^2 + b^2} \]

(b) Determine whether or not the filter is the same as that which would be designed using the impulse invariance method.

(a) If the impulse response of a continuous-time filter is \( h_a(t) \), its step response is

\[ s_a(t) = \int_{-\infty}^{t} h_a(\tau)d\tau \]

Therefore, because the Laplace transform of the step response is related to the system function \( H_a(s) \) as follows,

\[ S_a(s) = \frac{1}{s} H_a(s) \]

then

\[ S_a(s) = \frac{1}{s} \frac{s + a}{(s + a)^2 + b^2} \]
To design a digital filter using step invariance, we first perform a partial fraction expansion of \( S_a(s) \),

\[
S_a(s) = \frac{A_0}{s} + \frac{A_1}{s + a + jb} + \frac{A_2}{s + a - jb}
\]

where

\[
A_0 = \left[ sS_a(s) \right]_{s=0} = \frac{a}{a^2 + b^2}
\]

\[
A_1 = \left[ (s + a + jb)S_a(s) \right]_{s=0} = \frac{a}{(a^2 + b^2)(s + a - jb)}
\]

and

\[
A_2 = \left[ (s + a - jb)S_a(s) \right]_{s=0} = \frac{a}{(a^2 + b^2)(s + a + jb)}
\]

Therefore,

\[
S_a(s) = \frac{a}{a^2 + b^2} \times \frac{1}{s} - \frac{a + jb}{2(a^2 + b^2)(s + a - jb)} - \frac{a - jb}{2(a^2 + b^2)(s + a + jb)}
\]

Sampling \( s_a(t) \),

\[
s(n) = s_a(nT_c)
\]

and finding the z-transform of \( s(n) \) corresponds to the substitution

\[
\frac{1}{s - \alpha} \rightarrow \frac{1}{1 - e^{-\alpha T_c} z^{-1}}
\]

Thus, the z-transform of the step response is

\[
S(z) = \frac{a}{a^2 + b^2} \times \frac{1}{1 - z^{-1}} + \frac{a + jb}{2(a^2 + b^2)} \times \frac{1}{1 - e^{-(a+jb)T_c} z^{-1}} + \frac{a - jb}{2(a^2 + b^2)} \times \frac{1}{1 - e^{-(a-jb)T_c} z^{-1}}
\]

\[
= \frac{a}{a^2 + b^2} \times \frac{1}{1 - z^{-1}} + \frac{1}{a^2 + b^2} - \frac{a + \{a \cos(bT_c) + b \sin(bT_c)\}e^{-aT_c} z^{-1}}{1 - 2 \cos(bT_c) e^{-aT_c} z^{-1} + e^{-2aT_c} z^{-2}}
\]

The system function of the digital filter is then

\[
H(z) = (1 - z^{-1}) S(z) = \frac{a}{a^2 + b^2} \times \frac{1}{1 - z^{-1}} + \frac{1}{a^2 + b^2} - \frac{a + \{a \cos(bT_c) + b \sin(bT_c)\}e^{-aT_c} z^{-1}}{1 - 2 \cos(bT_c) e^{-aT_c} z^{-1} + e^{-2aT_c} z^{-2}}
\]

(b) Using impulse invariance, we see from Prob. 9.36 that the system function is

\[
H(z) = \frac{1/2}{1 - e^{-(a-jb)T_c} z^{-1}} + \frac{1/2}{1 - e^{(a+jb)T_c} z^{-1}} = \frac{1}{1 - e^{-aT_c} \cos(bT_c) z^{-1}} + \frac{1}{1 - e^{-aT_c} \cos(bT_c) z^{-1} + e^{-2aT_c} z^{-2}}
\]

Note that although \( H(z) \) has the same poles as the filter designed using step invariance, the system functions are not the same. Therefore, the two designs are not equivalent.

**9.38** Suppose that we would like to design a discrete-time low-pass filter by applying the impulse invariance method to a continuous-time Butterworth filter that has a magnitude-squared function

\[
|H_d(j\Omega)|^2 = \frac{1}{1 - (j\Omega/j\Omega_c)^{2N}}
\]
The specifications for the discrete-time filter are

\[ 1 - \delta_p \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq \omega \leq \omega_p \\
|H(e^{j\omega})| \leq \delta_s \quad \omega_s \leq \omega \leq \pi \]

Show that the design is not affected by the value of the sampling period that is used in the impulse invariance technique.

In the absence of aliasing, the impulse invariance method is a linear mapping from \( H_\omega(j\Omega) \) to \( H(e^{j\omega}) \) for \( |\omega| \leq \pi \). This mapping is

\[ H(e^{j\omega}) = H_\omega(j\Omega)|_{\omega=\Omega T} \quad |\omega| \leq \pi \]

Let us assume that there is no aliasing (this will be approximately true if the filter order is large enough). The required filter order is then

\[ N \geq \frac{\log d}{\log k} \]

where \( d \), the discrimination factor, is

\[ d = \left[ \frac{(1 - \delta_p)^2 - 1}{\delta_s^2 - 1} \right]^{1/2} \]

and \( k \), the selectivity factor, is

\[ k = \frac{\Omega_p}{\Omega_s} \]

Clearly, the discrimination factor does not depend on the sampling period \( T_s \). In addition, with \( \omega_p = \Omega_p T_s \) and \( \omega_s = \Omega_s T_s \), it follows that

\[ k = \frac{\omega_p / T_s}{\omega_s / T_s} = \frac{\omega_p}{\omega_s} \]

which does not depend on the sampling period. Therefore, the required filter order is independent of \( T_s \). Next, if we expand the system function of the Butterworth filter in a partial fraction expansion, we have

\[ H_\omega(s) = \sum_{k=0}^{N} \frac{A_k}{s - s_k} \]

where the poles, \( s_k \), are

\[ s_k = \Omega_s \exp\left\{ j \frac{(N + 1 + 2k)\pi}{2N} \right\} \quad k = 0, 1, \ldots, N - 1 \]

With impulse invariance, the system function of the discrete-time filter becomes

\[ H(z) = \sum_{k=0}^{N} \frac{A_k}{1 - e^{j\Omega_s T_s} z^{-1}} \]

and it follows that the poles of \( H(z) \) are at

\[ z = \exp\{s_k T_s\} = \exp\{\Omega_s T_s \theta_k\} \]

where

\[ \theta_k = \exp\left\{ j(N + 1 + 2k) \frac{\pi}{2N} \right\} \]

Because \( \omega_c = \Omega_s T_s \) is the 3-dB cutoff frequency of the low-pass filter in the discrete-time domain, it is fixed by the filter specifications. Therefore, the poles of \( H(z) \) will not be affected by the sampling period \( T_s \). For example, if we try to decrease \( T_s \) to reduce aliasing, this would require an increase in \( \Omega_s \) to preserve the cutoff frequency. Thus, it follows that the design is not affected by \( T_s \).

9.39 Use the impulse invariance method to design a low-pass digital Butterworth filter to meet the following specifications:

\[ 0.9 \leq |H(e^{j\omega})| \leq 1 \quad |\omega| \leq 0.2\pi \\
|H(e^{j\omega})| \leq 0.2 \quad 0.3\pi \leq \omega \leq \pi \]
In the absence of aliasing, the impulse invariance method is a linear mapping from \( H_a(j\Omega) \) to \( H(e^{j\omega}) \) for \( |\omega| \leq \pi \), which is given by
\[
H(e^{j\omega}) = H_a(j\Omega) \bigg|_{\omega = \Omega T}, \quad |\omega| \leq \pi
\]
Therefore, in order to simplify the design, we will assume that there is no aliasing and then, after the design is completed, check to see that the filter satisfies the given specifications. Because the parameter \( T \) does not enter into the design using the impulse invariance method (see Prob. 9.38), for convenience we will set \( T = 1 \).

The first step, then, is to design an analog Butterworth filter according to the following specifications:

\[
\begin{align*}
0.9 & \leq |H_a(j\Omega)| \leq 1 & 0 & \leq |\Omega| \leq 0.2\pi \\
|H_a(j\Omega)| & \leq 0.2 & 0.3\pi & \leq \Omega
\end{align*}
\]

To determine the filter order, we compute the discrimination factor,
\[
d = \left[ \frac{(1 - \delta_p)^2 - 1}{\delta_s^2 - 1} \right]^{1/2} = 0.0989
\]
and the selectivity factor,
\[
k = \frac{0.2\pi}{0.3\pi} = 0.667
\]
Thus, we have
\[
N = \log d / \log k = 5.71
\]
which, when rounded up, gives \( N = 6 \).

For the 3-dB cutoff frequency of the Butterworth filter, we will select \( \Omega_c \) so that
\[
|H_a(j\Omega)|^2 \bigg|_{\Omega = 0.2\pi} = (0.9)^2
\]
that is, so that the Butterworth filter satisfies the passband specifications exactly (this will provide for some allowance for aliasing in the stopband). With
\[
|H_a(j\Omega)|^2 \bigg|_{\Omega = 0.2\pi} = \frac{1}{1 + (0.2\pi / \Omega_c)^{12}} = 0.81
\]
we have
\[
1 = 0.81 \left[ 1 + \left( \frac{0.2\pi}{\Omega_c} \right)^{12} \right]
\]
or
\[
0.19 = 0.81 \left( \frac{0.2\pi}{\Omega_c} \right)^{12}
\]
which gives
\[
\Omega_c = 0.2\pi \left( \frac{0.81}{0.19} \right)^{1/12} = 0.7090
\]
Therefore, the magnitude of the frequency response squared is
\[
|H_a(j\Omega)|^2 = \frac{1}{1 + (\Omega / 0.7090)^{12}}
\]
and the 12 poles of
\[
H_a(s)H_a(-s) = \frac{1}{1 + (s/j\Omega_c)^{12}}
\]
lie on a circle of radius \( \Omega_c = 0.7090 \), at angles
\[
\theta_k = \frac{(2k + 1)\pi}{12}, \quad k = 0, 1, 2, \ldots, 11
\]
Thus, the poles of the Butterworth filter are the three complex conjugate pole pairs of \( H_o(s)H_o(-s) \) that are in the left-half \( s \)-plane:

\[
\begin{align*}
    s_0 &= s_1^* = 0.7090e^{j7\pi/12} \\
    s_2 &= s_3^* = 0.7090e^{j9\pi/12} \\
    s_4 &= s_5^* = 0.7090e^{j11\pi/12}
\end{align*}
\]

Therefore, with

\[
H_o(s) = \prod_{k=0}^{5} \frac{-s_k}{s - s_k}
\]

forming second-order polynomials from each conjugate pole pair, we have

\[
H_o(s) = \frac{(0.7090)^6}{(s^2 + 0.3670s + 0.5027)(s^2 + 1.0027s + 0.5027)(s^2 + 1.3697s + 0.5027)}
\]

The next steps, which are algebraically very tedious, are to perform a partial fraction expansion of \( H_o(s) \),

\[
H_o(s) = \sum_{k=1}^{6} \frac{A_k}{s - 0.7090e^{j(2k+1)\pi/12}}
\]

perform the transformation

\[
\frac{1}{s - s_k} \rightarrow \frac{1}{1 - e^{j\theta} z^{-1}}
\]

and then recombine the terms. The result is

\[
H(z) = \frac{0.0007z^{-1} + 0.0105z^{-2} + 0.0168z^{-3} + 0.0042z^{-4} + 0.0001z^{-5}}{1 - 3.3431z^{-1} + 5.0150z^{-2} - 4.2153z^{-3} + 2.0703z^{-4} - 0.5593z^{-5} + 0.0646z^{-6}}
\]

The magnitude of the frequency response in decibels is plotted in the following figure.
As a final check on the design, evaluating $H(e^{j\omega})$ at $\omega = 0.2\pi$ and $\omega = 0.3\pi$, we find that

$$\left|H(e^{j\omega})\right|_{\omega=0.2\pi} = 0.9219$$

and

$$\left|H(e^{j\omega})\right|_{\omega=0.3\pi} = 0.2045$$

Therefore, the filter exceeds the passband specifications and comes close to meeting the stopband specifications. If this filter is unacceptable, the design could be modified by decreasing $Q$, to improve the stopband performance.

**9.40** Repeat Prob. 9.39 using the bilinear transformation.

Using the bilinear transformation to design a Butterworth filter according to the specifications given in Prob. 9.39, we first use the transformation

$$\Omega = \frac{2}{T_s} \tan \left( \frac{\omega}{2} \right)$$

to map the passband and stopband frequencies of the digital filter to the cutoff frequencies of the analog filter. With $T_s = 2$, we have

$$\Omega_p = \tan \left( \frac{\omega_p}{2} \right) = \tan(0.1\pi) = 0.3249$$

and

$$\Omega_s = \tan \left( \frac{\omega_s}{2} \right) = \tan(0.15\pi) = 0.5095$$

As we found in Prob. 9.39, the required filter order is $N = 6$. For the 3-dB cutoff frequency of the analog Butterworth filter, we may choose any frequency in the range

$$0.3667 \leq \Omega_s \leq 0.3910$$

If we select $\Omega_s = 0.3667$, the passband specifications will be met exactly, and the stopband specifications will be exceeded. Conversely, if we set $\Omega_s = 0.3910$, the stopband specifications will be met exactly, and the passband specifications exceeded. Picking a frequency between the two extremes will produce an improvement in both bands. Because the stopband deviation is twice that of the deviation in the passband, we will set $\Omega_s = 0.3667$ in order to improve the stopband performance. From Table 9-4, we find the coefficients in the system function of a sixth-order normalized ($\Omega_s = 1$) Butterworth filter to be

$$H_6(s) = \frac{1}{s^6 + 3.8637s^5 + 7.4641s^4 + 9.1416s^3 + 7.4641s^2 + 3.8637s + 1}$$

To obtain a Butterworth filter with a cutoff frequency $\Omega_s = 0.3666$, we perform the low-pass-to-low-pass transformation

$$s \rightarrow \frac{s}{0.3667}$$
This gives
\[ H_a(s) = \frac{(0.3666)^6}{s^6 + 1.4165s^5 + 1.0033s^4 + 0.4505s^3 + 0.1349s^2 + 0.0256s + 0.0024} \]
Finally, we apply the bilinear transformation
\[ s = \frac{1 - z^{-1}}{1 + z^{-1}} \]
which yields the digital filter
\[ H(z) = \frac{0.0006 + 0.0036z^{-1} + 0.0090z^{-2} + 0.0120z^{-3} + 0.0090z^{-4} + 0.0036z^{-5} + 0.0006z^{-6}}{1 - 3.2942z^{-1} + 4.8985z^{-2} - 4.0857z^{-3} + 1.9932z^{-4} - 0.5353z^{-5} + 0.0615z^{-6}} \]
At the passband cutoff frequency, \( \omega_p = 0.2\pi \), the magnitude of the frequency response is
\[ |H(e^{j\omega})|_{\omega=0.2\pi} = 0.0905 \]
and at the stopband cutoff frequency, \( \omega_s = 0.3\pi \), the magnitude of the frequency response is
\[ |H(e^{j\omega})|_{\omega=0.3\pi} = 0.1392 \]
Therefore, this filter meets the given specifications.

9.41 Use the bilinear transformation to design a first-order low-pass Butterworth filter that has a 3-dB cutoff frequency \( \omega_c = 0.2\pi \).

If a digital low-pass filter is to have a 3-dB cutoff frequency at \( \omega_c = 0.2\pi \), the analog Butterworth filter should have a 3-dB cutoff frequency
\[ \Omega_c = \tan\left(\frac{\omega_c}{2}\right) = \tan(0.1\pi) = 0.3249 \]
For a first-order Butterworth filter,
\[ H_a(s) = \frac{s + \Omega_c}{s + \Omega_c} \]
Therefore, the system function is
\[ H_a(s) = \frac{\Omega_c}{s + \Omega_c} \]
With \( \Omega_c = 0.3249 \), applying the bilinear transformation
\[ s = \frac{1 - z^{-1}}{1 + z^{-1}} \]
we have
\[ H(z) = \frac{0.3249(1 + z^{-1})}{1 - z^{-1} + 0.3249} = \frac{0.3249(1 + z^{-1})}{1 - 0.5095z^{-1}} \]

9.42 A second-order continuous-time filter has a system function
\[ H_a(s) = \frac{1}{s - a} + \frac{1}{s - b} \]
where \( a < 0 \) and \( b < 0 \) are real.

(a) Determine the locations of the poles and zeros of \( H(z) \) if the filter is designed using the bilinear transformation with \( T_s = 2 \).

(b) Repeat part (a) for the impulse invariance technique, again with \( T_s = 2 \).
(a) The bilinear transformation is defined by the mapping

\[ s = \frac{1 - z^{-1}}{1 + z^{-1}} \]

Therefore, for the given filter, we have

\[
H(z) = \frac{1}{1 - z^{-1} - a} + \frac{i}{1 - z^{-1} - b} = \frac{1 + z^{-1}}{(1 - a) - z^{-1}} + \frac{1 + z^{-1}}{(1 - b) - z^{-1}}
\]

which has poles at

\[
z_1 = \frac{1}{1 - a} \quad \text{and} \quad z_2 = \frac{1}{1 - b}
\]

To find the zeros, it is necessary to combine the two terms in the system function over a common denominator. Doing this, we have

\[
H(z) = \frac{[(1 - b) - z^{-1}][(1 + z^{-1}) + (1 - a) - z^{-1}](1 + z^{-1})}{[(1 - a) - z^{-1}][(1 - b) - z^{-1}]}
\]

Finding the roots of the numerator may be facilitated by noting that \(H_a(s)\) has a zero at \(s = \infty\), which gets mapped to \(z = -1\) with the bilinear transformation. Therefore, one of the factors of the numerator is \((1 + z^{-1})\). Dividing the numerator by this factor, we obtain the second factor, which is \((2 - a - b) - 2z^{-2}\). Thus, \(H(z)\) has zeros at

\[
z_1 = -1 \quad \text{and} \quad z_2 = \frac{2}{2 - a - b}
\]

(b) With the impulse invariance technique, for first-order poles, the mapping is

\[
\frac{1}{s - s_k} \Rightarrow \frac{1}{1 - e^{\alpha T} z^{-1}}
\]

Therefore, with \(T_s = 2\) we have

\[
H(z) = \frac{1}{1 - e^{2\alpha} z^{-1}} + \frac{1}{1 - e^{2b} z^{-1}} = \frac{2 - e^{2\alpha} z^{-1} - e^{2b} z^{-1}}{(1 - e^{2\alpha} z^{-1})(1 - e^{2b} z^{-1})}
\]

which has two poles at

\[
z_1 = e^{2\alpha} \quad \text{and} \quad z_2 = e^{2b}
\]

and only one zero, which is located at

\[
z_0 = \frac{1}{2} (e^{2\alpha} + e^{2b})
\]

9.43 The system function of a digital filter is

\[
H(z) = \sum_{k=1}^{p} \frac{A_k}{1 - a_k z^{-1}}
\]

(a) If this filter was designed using impulse invariance with \(T_s = 2\), find the system function, \(H_d(s)\), of an analog filter that could have been the analog filter prototype. Is your answer unique?

(b) Repeat part (a) assuming that the bilinear transformation was used with \(T_s = 2\).

(a) Because \(H(z)\) is expanded in a partial fraction expansion, the poles at \(z = a_k\) in \(H(z)\) are mapped from poles in \(H_s(s)\) according to the mapping

\[
a_k = e^{\alpha T_s}
\]

Therefore, if \(T_s = 2\),

\[
s_k = \frac{1}{2} \ln a_k
\]
and one possible analog filter prototype is

\[ H_a(s) = \sum_{k=1}^{p} \frac{A_k}{s - \frac{1}{2} \ln \alpha_k} \]

Because the mapping from the \( s \)-plane to the \( z \)-plane is not one to one, this answer is not unique. Specifically, note that we may also write

\[ \alpha_k = e^{i \pi T_s + j \pi} \]

Therefore, with \( T_s = 2 \), we may also have

\[ s_k = \frac{1}{2} \ln \alpha_k + j \pi \]

and another possible analog filter prototype is

\[ H_a(s) = \sum_{k=1}^{p} \frac{A_k}{s - \left( \frac{1}{2} \ln \alpha_k - j \pi \right)} \]

(b) With the bilinear transformation, because the mapping from the \( s \)-plane to the \( z \)-plane is a one-to-one mapping, with \( T_s = 2 \),

\[ z = \frac{1 + s}{1 - s} \]

and the analog filter prototype that is mapped to \( H(z) \) is unique and given by

\[ H_a(s) = \sum_{k=1}^{p} A_k \frac{1}{1 - \alpha_k} \frac{1 - s}{1 + s} = \sum_{k=1}^{p} \frac{A_k (1 + s)}{(1 - \alpha_k) + (1 + \alpha_k)s} \]

9.44 A continuous-time system is called an *integrator* if the response of the system \( y_c(t) \) to an input \( x_c(t) \) is

\[ y_c(t) = \int_{-\infty}^{t} x_c(\tau) d\tau \]

The system function for an integrator is

\[ H_a(s) = \frac{1}{s} \]

(a) Design a discrete-time “integrator” using the bilinear transformation, and find the difference equation relating the input \( x(n) \) to the output \( y(n) \) of the discrete-time system.

(b) Find the frequency response of the discrete-time integrator found in part (a), and determine whether or not this system is a good approximation to the continuous-time system.

(a) With a system function \( H_a(s) = 1/s \), the bilinear transformation produces a discrete-time filter with a system function

\[ H(z) = \frac{T_s}{2} \frac{1 + z^{-1}}{1 - z^{-1}} \]

The unit sample response of this filter is

\[ h(n) = \frac{T_s}{2} [u(n) + u(n - 1)] \]

and the difference equation relating the output \( y(n) \) to the input \( x(n) \) is

\[ y(n) = y(n - 1) + \frac{T_s}{2} [x(n) + x(n - 1)] \]
(b) Because the frequency response of the continuous-time system, \( H_c(j\Omega) = 1/j\Omega \), is related to the discrete-time filter through the mapping

\[
\Omega = \frac{2}{T_c} \tan \left( \frac{\omega}{2} \right)
\]

the frequency response of the discrete-time system is

\[
H(e^{j\omega}) = H_c(j\Omega) \bigg|_{\Omega = \frac{2}{T_c} \tan \left( \frac{\omega}{2} \right)} = \frac{T_s}{2} \cot \left( \frac{\omega}{2} \right)
\]

Note that because \( H(e^{j\omega}) \) goes to zero at \( \omega = \pi \), then \( H(e^{j\omega}) \) will not be a good approximation to \( H_c(j\Omega) = 1/j\Omega \) except for low frequencies. However, if \( \omega \ll 1 \), using the expansion

\[
\cos x \approx 1 - \frac{1}{2} x^2 \quad x \ll 1
\]

and

\[
\sin x \approx x \quad x \ll 1
\]

we have

\[
\cot \left( \frac{\omega}{2} \right) \approx \frac{\cos(\omega/2)}{\sin(\omega/2)} \approx \frac{1 - \frac{1}{8}(\omega/2)^2}{\omega/2}
\]

and we have, for the frequency response,

\[
H(e^{j\omega}) = \frac{T_s}{2} \frac{\cos(\omega/2)}{\sin(\omega/2)} \approx \frac{T_s}{j\omega} \left( 1 - \frac{1}{8} \omega^2 \right)
\]

Therefore, for small \( \omega \)

\[
H(e^{j\omega}) \approx T_s H_c(j\Omega)
\]

### 9.45 Let \( H_c(j\Omega) \) be an \( N \)th-order low-pass Butterworth filter with a 3-dB cutoff frequency \( \Omega_c \).

(a) Show that \( H_c(s) \) may be transformed into an \( N \)th-order high-pass Butterworth filter by adding \( N \) zeros at \( s = 0 \) and scaling the gain.

(b) What is the relationship between the corresponding digital low-pass and high-pass Butterworth filters that are designed using the bilinear transformation?

(a) For an \( N \)th-order low-pass Butterworth with a system function \( H_c(s) \),

\[
H_c(s)H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^N}
\]

Adding \( N \) zeros to \( H_c(s) \) at \( s = 0 \), we have

\[
H_c(s)H_c(-s) = \frac{(-1)^N s^{2N}}{1 + (s/j\Omega_c)^N}
\]

Multiplying numerator and denominator by \((j\Omega_c/s)^{2N}\) yields

\[
H_c(s)H_c(-s) = \frac{(j\Omega_c)^{2N}}{1 + (j\Omega_c/s)^{2N}}
\]

which corresponds to a magnitude-squared frequency response

\[
|H_c(j\Omega)|^2 = \frac{\Omega_c^{2N}}{1 + (\Omega_c/\Omega)^{2N}}
\]

Scaling \( H_c(j\Omega) \) by \( \Omega^{-N} \) results in a filter that has a frequency response \( G_c(j\Omega) \) with a squared magnitude

\[
|G_c(j\Omega)|^2 = \frac{1}{\Omega_c^{2N}} |H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega_c/\Omega)^{2N}}
\]

which is a high-pass filter with a cutoff frequency \( \Omega_c \). Specifically, note that \(|G_c(j\Omega)|^2\) is equal to zero at \( \Omega = 0 \), and that \(|G_c(j\Omega)|^2 \rightarrow 1\) as \( \Omega \rightarrow \infty \).
Applying the bilinear transformation to a low-pass Butterworth filter, we have

$$H(z)H(z^{-1}) = H_d(s)H_d(-s)\big|_{s = \frac{1}{1 + z^{-1}}}$$

$$= \frac{1}{1 + (j\Omega_c)^{-2N} \frac{1 - z^{-1}}{1 + z^{-1}}}$$

$$= \frac{(1 + z^{-1})^2N (j\Omega_c)^{2N}}{(1 - z^{-1})^{2N} + (1 + z^{-1})^{2N}}$$

For the high-pass filter, on the other hand, we have

$$G(z)G(z^{-1}) = G_d(s)G_d(-s)\big|_{s = \frac{1}{1 + z^{-1}}}$$

$$= \frac{1}{1 + (j\Omega_c)^{2N} \frac{1 - z^{-1}}{1 + z^{-1}}}$$

$$= \frac{(1 - z^{-1})^{2N}}{(1 - z^{-1})^{2N} + (j\Omega_c)^{2N}(1 + z^{-1})^{2N}}$$

Therefore, we see that the poles of the low-pass digital Butterworth filter are the same as those of the high-pass digital Butterworth filter. The zeros, however, which are at $z = -1$ in the case of the low-pass filter, are at $z = 1$ in the high-pass filter. Thus, except for a difference in the gain, the high-pass digital Butterworth filter may be derived from the low-pass filter by flipping the $N$ zeros in $H(z)$ at $z = -1$ to $z = 1$.

9.46 The impulse invariance method and the bilinear transformation are two filter design techniques that preserve stability of the analog filter by mapping poles in the left-half $s$-plane to poles inside the unit circle in the $z$-plane. An analog filter is minimum phase if all of its poles and zeros are in the left-half $s$-plane.

(a) Determine whether or not a minimum phase analog filter is mapped to a minimum phase discrete-time system using the impulse invariance method.

(b) Repeat part (a) for the bilinear transformation.

(a) With impulse invariance, an analog filter with a system function

$$H_a(s) = \sum_{k=1}^{p} \frac{A_k}{s - s_k}$$

will be mapped to a digital filter with a system function

$$H(z) = \sum_{k=1}^{p} \frac{A_k}{1 - e^{s_k} z^{-1}}$$

Rewriting this system function as a ratio of polynomials, it follows that the locations of the zeros of $H(z)$ will depend on the locations of poles as well as the zeros of $H_a(s)$, and there is no way to guarantee that the zeros lie inside the unit circle. A simple example showing that a minimum phase continuous-time filter will not necessarily be mapped to a minimum phase discrete-time filter is the following:

$$H_a(s) = \frac{s + 8}{(s + 1)(s + 2)} = \frac{7}{s + 1} - \frac{6}{s + 2}$$

Using the impulse invariance method with $T_s = 1$, we have

$$H(z) = \frac{7}{1 - e^{-1} z^{-1}} - \frac{6}{1 - e^{-2} z^{-1}} = \frac{1 + (6e^{-1} - 7e^{-2})z^{-1}}{(1 - e^{-1} z^{-1})(1 - e^{-2} z^{-1})}$$

which has a zero at

$$z = -(6e^{-1} - 7e^{-2}) \approx -1.256$$

Therefore, although $H_a(s)$ is minimum phase, $H(z)$ is not.
(b) The mapping between the s-plane and the z-plane with the bilinear transformation is defined by

\[ z = \frac{1 + (T_s/2)s}{1 - (T_s/2)s} \]

Therefore, a pole or a zero at \( s = s_k \) becomes a pole or a zero at

\[ z_k = \frac{1 + (T_s/2)s_k}{1 - (T_s/2)s_k} \]

If \( H_a(s) \) is minimum phase, the poles and zeros of \( H_a(s) \) are in the left-half s-plane. In other words, if \( H_a(s) \) has a pole or a zero at \( s = s_k \), where \( s_k = \sigma_k + j\Omega_k \),

\[ \sigma_k < 0 \]

Therefore,

\[ |z_k|^2 = \left| \frac{1 + (T_s/2)s_k}{1 - (T_s/2)s_k} \right|^2 = \frac{|(2/T_s) + \sigma_k|^2}{|2/T_s - \sigma_k|^2} = \frac{|(2/T_s) + \sigma_k|^2 + \Omega_k^2}{|2/T_s - \sigma_k|^2 + \Omega_k^2} < 1 \]

and it follows that a pole or a zero in the left-half s-plane will be mapped to a pole or a zero inside the unit circle in the z-plane (i.e., \( H(z) \) is minimum phase).

9.47 The system function of a continuous-time filter \( H_a(s) \) of order \( N \geq 2 \) may be expressed as a cascade of two lower-order systems:

\[ H_a(s) = H_{a1}(s)H_{a2}(s) \]

Therefore, a digital filter may either be designed by applying a transformation directly to \( H_a(s) \) or by individually transforming \( H_{a1}(s) \) and \( H_{a2}(s) \) into \( H_1(z) \) and \( H_2(z) \), respectively, and then realizing \( H(z) \) as the cascade:

\[ H(z) = H_1(z)H_2(z) \]

(a) If \( H_1(z) \) and \( H_2(z) \) are designed from \( H_{a1}(s) \) and \( H_{a2}(s) \) using the impulse invariance technique, compare the cascade \( H(z) = H_1(z)H_2(z) \) with the filter that is designed by using the impulse invariance technique directly on \( H_a(s) \).

(b) Repeat part (a) for the bilinear transformation.

(a) Due to sampling, aliasing occurs when designing a digital filter using the impulse invariance method. Because the operations of sampling and convolution do not commute, the filter designed by using impulse invariance on \( H_a(s) \) will not be the same as the filter designed by cascading the two filters that are designed using impulse invariance on \( H_{a1}(s) \) and \( H_{a2}(s) \). In other words, if

\[ h_a(t) = h_{a1}(t) * h_{a2}(t) \]

then

\[ h(n) \neq h_1(n) * h_2(n) \]

where \( h(n) = h_a(nT_s) \), \( h_1(n) = h_{a1}(nT_s) \), and \( h_2(n) = h_{a2}(nT_s) \). As an example, consider the continuous-time filter that has a system function

\[ H_a(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \]

Using the impulse invariance technique on \( H_a(s) \) with \( T_s = 1 \), we have

\[ H(z) = \frac{1}{1 - e^{-1}z^{-1}} - \frac{1}{1 - e^{-2}z^{-1}} = \frac{(e^{-1} - e^{-2})z^{-1}}{(1 - e^{-1}z^{-1})(1 - e^{-2}z^{-1})} \]

On the other hand, writing \( H_a(s) \) as a cascade of two first-order systems,

\[ H_a(s) = \frac{1}{s+1} \cdot \frac{1}{s+2} \]

and using the impulse invariance method on each of these systems with \( T_s = 1 \), we have

\[ H(z) = \frac{1}{1 - e^{-1}z^{-1}} \cdot \frac{1}{1 - e^{-2}z^{-1}} \]

which is not the same as the previous filter.
(b) With the bilinear transformation, \( T_s = 2 \)

\[
H(z) = H_n \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) = H_n \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) H_n \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) = H_1(z)H_2(z)
\]

and the two designs are the same.

9.48 What are the properties of the \( s \)-plane-to-\( z \)-plane mapping defined by

\[
s = \frac{1 + z^{-1}}{1 - z^{-1}}
\]

and what might this mapping be used for?

This mapping is very similar to the bilinear transformation which, with \( T_s = 2 \), is

\[
s = \frac{1 - z^{-1}}{1 + z^{-1}}
\]

In fact, this mapping may be considered to be a cascade of two mappings. The first is the bilinear transformation, and the second is one that replaces \( z \) with \(-z\),

\[
z' = -z
\]

This mapping reflects points in the \( z \)-plane about the origin and, for points on the unit circle, corresponds to a shift of \( 180^\circ \):

\[
H(z')|_{z'=e^{j\omega}} = H(-z)|_{z=e^{j\omega}} = H(-e^{j\omega}) = H(e^{j(\omega + \pi)})
\]

Therefore, this mapping has the same properties as the bilinear transformation, except that the \( j\Omega \) axis is mapped onto the unit circle by the mapping

\[
\omega = 2 \arctan \left( \frac{\Omega T_s}{2} \right) + \pi
\]

Because the unit circle is rotated by \( 180^\circ \), this mapping may be used to map low-pass analog filters into high-pass digital filters, and high-pass analog filters into low-pass digital filters.

**Least-Squares Filter Design**

9.49 Suppose that the desired unit sample response of a linear shift-invariant system is

\[
h_d(n) = 3 \left( \frac{1}{2} \right)^n u(n)
\]

Use the Padé approximation method to find the parameters of a filter with a system function

\[
H(z) = \frac{b(0) + b(1)z^{-1}}{1 + a(1)z^{-1}}
\]

that approximates this unit sample response.

Using the Padé approximation method, with \( p = q = 1 \), we want to solve the following set of linear equations for \( b(0) \), \( b(1) \), and \( a(1) \):

\[
\begin{bmatrix}
    h_d(0) & 0 \\
    h_d(1) & h_d(0) \\
    h_d(2) & h_d(1)
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a(1)
\end{bmatrix}
=
\begin{bmatrix}
    b(0) \\
    b(1) \\
    0
\end{bmatrix}
\]

Substituting the given values for \( h_d(n) \), we have

\[
\begin{bmatrix}
    3 & 0 \\
    3/2 & 3 \\
    3/4 & 3/2
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a(1)
\end{bmatrix}
=
\begin{bmatrix}
    b(0) \\
    b(1) \\
    0
\end{bmatrix}
\]
Using the last equation, we may easily solve for \( a(1) \),
\[
\frac{1}{4} + \frac{3}{2} a(1) = 0
\]
or
\[
a(1) = -\frac{1}{3}
\]
Having found \( a(1) \), we may solve for \( b(0) \) and \( h(1) \) using the first two equations
\[
\begin{bmatrix}
3 & 0 \\
\frac{3}{2} & 3
\end{bmatrix}
\begin{bmatrix}
1 \\
-\frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
b(0) \\
b(1)
\end{bmatrix}
\]
or
\[
h(0) = 3 \quad b(1) = 0
\]
Therefore, we have
\[
H(z) = \frac{3}{1 - 0.5z^{-1}}
\]
Notice that the unit sample response corresponding to this system exactly matches the given unit sample response. In general, however, this will not be true. A perfect match depends on \( h_{d}(n) \) being the inverse \( z \)-transform of a rational function of \( z \), and it depends upon an appropriate choice for the order of the Padé approximation (the number of poles and zeros).

**9.50**

Let the first three values of the unit sample response of a desired causal filter be \( h_{d}(0) = 3 \), \( h_{d}(1) = \frac{1}{4} \), and \( h_{d}(2) = \frac{1}{16} \).

(a) Using the Padé approximation method, find the coefficients of a second-order all-pole filter that has a unit sample response \( h(n) \), such that \( h(n) = h_{d}(n) \) for \( n = 0, 1, 2 \).

(b) Repeat part (a) for a filter that has one pole and one zero.

(c) Repeat part (a) for an FIR filter that has two zeros.

(a) For a second-order all-pole filter, the equations for the Padé approximation are
\[
\begin{bmatrix}
3 & 0 & 0 \\
\frac{3}{4} & 3 & 0 \\
\frac{1}{16} & \frac{3}{4} & 3
\end{bmatrix}
\begin{bmatrix}
1 \\
\frac{1}{4} a(1) \\
\frac{1}{16} a(2)
\end{bmatrix}
= \begin{bmatrix}
b(0) \\
b(1) \\
b(2)
\end{bmatrix}
\]
which, with the given values for \( h_{d}(n) \) become
\[
\begin{bmatrix}
3 & 0 & 0 \\
\frac{3}{4} & 3 & 0 \\
\frac{1}{16} & \frac{3}{4} & 3
\end{bmatrix}
\begin{bmatrix}
1 \\
\frac{1}{4} a(1) \\
\frac{1}{16} a(2)
\end{bmatrix}
= \begin{bmatrix}
b(0) \\
b(1) \\
b(2)
\end{bmatrix}
\]
From the last two equations, we have
\[
\begin{bmatrix}
3 \\
\frac{3}{4} \\
\frac{1}{16}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{4} \\
\frac{1}{16}
\end{bmatrix}
= \begin{bmatrix}
3 \\
\frac{3}{4} \\
\frac{1}{16}
\end{bmatrix}
\]
Solving for \( a(1) \) and \( a(2) \), we find
\[
a(1) = -\frac{1}{12} \quad a(2) = -\frac{1}{12}
\]
Then, using the first equation, we have
\[
h(0) = 3
\]
Thus, the system function of the filter is
\[
H(z) = \frac{3}{1 - \frac{1}{12} z^{-1} - \frac{1}{12} z^{-2}}
\]
(b) Using a first-order system to match the given values of $h_d(n)$, we have

$$H(z) = \frac{b(0) + b(1)z^{-1}}{1 + a(1)z^{-1}}$$

the equations that we must solve are as follows,

$$\begin{bmatrix}
    h_d(0) & 0 \\
    h_d(1) & h_d(0) \\
    h_d(2) & h_d(1)
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a(1)
\end{bmatrix}
= \begin{bmatrix}
    b(0) \\
    b(1) \\
    0
\end{bmatrix}$$

or, using the values for $h_d(n)$,

$$\begin{bmatrix}
    3 & 0 \\
    \frac{1}{4} & 3 \\
    \frac{1}{16} & \frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a(1)
\end{bmatrix}
= \begin{bmatrix}
    b(0) \\
    b(1) \\
    0
\end{bmatrix}$$

We may solve for $a(1)$ using the last equation.

$$\frac{1}{16} + \frac{1}{4}a(1) = 0$$

or

$$a(1) = -\frac{1}{4}$$

Next, we solve for $b(0)$ and $b(1)$ using the first two equations,

$$\begin{bmatrix}
    3 & 0 \\
    \frac{1}{4} & 3
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a(1)
\end{bmatrix}
= \begin{bmatrix}
    b(0) \\
    b(1)
\end{bmatrix}$$

which gives

$$\begin{bmatrix}
    b(0) \\
    b(1)
\end{bmatrix}
= \begin{bmatrix}
    3 & 0 \\
    \frac{1}{4} & 3
\end{bmatrix}
\begin{bmatrix}
    1 \\
    -\frac{1}{4}
\end{bmatrix}
= \begin{bmatrix}
    3 \\
    -\frac{1}{2}
\end{bmatrix}$$

Thus, the system function is

$$H(z) = \frac{3 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

(c) For an FIR filter, the solution is trivial:

$$H(z) = b(0) + b(1)z^{-1} + b(2)z^{-2} = h_d(0) + h_d(1)z^{-1} + h_d(2)z^{-2} = 3 + \frac{1}{4}z^{-1} + \frac{1}{16}z^{-2}$$

9.51 Find the least-squares FIR inverse filter of length 3 for the system that has a unit sample response

$$g(n) = \begin{cases}
    2 & n = 0 \\
    1 & n = 1 \\
    0 & \text{else}
\end{cases}$$

Also, find the least-squares error,

$$E = \sum_{n=0}^{\infty} e^2(n)$$

for this least-squares inverse filter.

To find the least-squares inverse, we need to solve the linear equations

$$\sum_{l=0}^{N-1} h(l)r_s(k-l) = \begin{cases}
    g(0) & k = 0 \\
    0 & k = 1, 2, \ldots, N - 1
\end{cases}$$
is the deterministic autocorrelation of \( g(n) \). With \( N = 3 \), these equations may be written in matrix form as follows,

\[
\begin{bmatrix}
r_g(0) & r_g(1) & r_g(2) \\
r_g(1) & r_g(0) & r_g(1) \\
r_g(2) & r_g(1) & r_g(0)
\end{bmatrix}
\begin{bmatrix}
h(0) \\
h(1) \\
h(2)
\end{bmatrix}
= \begin{bmatrix} g(0) \\ 0 \\ 0 \end{bmatrix}
\]

For the given sequence \( g(n) \), we compute the autocorrelation sequence as follows,

\[
\begin{align*}
r_g(0) &= \sum_{n=0}^{\infty} g^2(n) = 5 \\
r_g(1) &= \sum_{n=0}^{\infty} g(n)g(n-1) = 2 \\
r_g(2) &= \sum_{n=0}^{\infty} g(n)g(n-2) = 0
\end{align*}
\]

Therefore, the linear equations become

\[
\begin{bmatrix}
5 & 2 & 0 \\
2 & 5 & 2 \\
0 & 2 & 5
\end{bmatrix}
\begin{bmatrix}
h(0) \\
h(1) \\
h(2)
\end{bmatrix}
= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}
\]

and the solution is

\[
h(n) = \begin{cases} 
0.494 & n = 0 \\
-0.235 & n = 1 \\
0.094 & n = 2 \\
0 & \text{else}
\end{cases}
\]

Performing the convolution of \( h(n) \) with \( g(n) \), we have

\[
g(n) * h(n) = \begin{cases} 
0.988 & n = 0 \\
0.023 & n = 1 \\
-0.047 & n = 2 \\
0.094 & n = 3
\end{cases}
\]

From this sequence, we may evaluate the squared error,

\[
\mathcal{E} = \sum_{n=0}^{\infty} [g(n) * h(n) - \delta(n)]^2 = 0.0118
\]

**9.52** Find the FIR least-squares inverse filter of length \( N \) for the system having a unit sample response

\[
g(n) = \delta(n) - \alpha \delta(n - 1)
\]

where \( \alpha \) is an arbitrary real number.

Before we begin, note that if \( |\alpha| > 1 \), \( G(z) \) has a zero that is outside the unit circle. In this case, \( G(z) \) is not minimum phase, and the inverse filter \( 1/G(z) \) cannot be both causal and stable. However, if \( |\alpha| < 1 \),

\[
G^{-1}(z) = \frac{1}{G(z)} = \frac{1}{1 - \alpha z^{-1}}
\]

and the inverse filter is

\[
g^{-1}(n) = \alpha^n u(n)
\]
We begin by finding the least-squares inverse of length $N = 2$. The autocorrelation sequence $r_x(k)$ is

$$\begin{align*}
r_x(k) &= \begin{cases} 
1 + \alpha^2 & k = 0 \\
-\alpha & k = \pm 1 \\
0 & \text{else}
\end{cases}
\end{align*}$$

Therefore, the linear equations that we must solve are

$$\begin{bmatrix} 1 + \alpha^2 & -\alpha \\
-\alpha & 1 + \alpha^2 \end{bmatrix} \begin{bmatrix} h(0) \\
h(1) \end{bmatrix} = \begin{bmatrix} 1 \\
0 \end{bmatrix}$$

The solution for $h(0)$ and $h(1)$ is easily seen to be

$$\begin{align*}
h(0) &= \frac{1 + \alpha^2}{1 + \alpha^2 + \alpha^4} \\
h(1) &= \frac{\alpha}{1 + \alpha^2 + \alpha^4}
\end{align*}$$

The system function of this least-squares inverse filter is

$$H(z) = \frac{1 + \alpha^2}{1 + \alpha^2 + \alpha^4} + \frac{\alpha}{1 + \alpha^2 + \alpha^4} z^{-1} = \frac{1 + \alpha^2}{1 + \alpha^2 + \alpha^4} \left( 1 + \frac{\alpha}{1 + \alpha^2} z^{-1} \right)$$

which has a zero at

$$z_0 = -\frac{\alpha}{1 + \alpha^2}$$

Note that because

$$|z_0| = \frac{|\alpha|}{1 + \alpha^2} = \frac{1}{|\alpha + \alpha^{-1}|} < 1$$

the zero of $H(z)$ is inside the unit circle, and $H(z)$ is minimum phase, regardless of whether the zero of $G(z)$ is inside or outside the unit circle.

Let us now look at the least-squares inverse, $h_N(n)$, of length $N$. In this case, the linear equations have the form

$$\begin{bmatrix} 1 + \alpha^2 & -\alpha & 0 & \cdots & 0 \\
-\alpha & 1 + \alpha^2 & -\alpha & \cdots & 0 \\
0 & -\alpha & 1 + \alpha^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 + \alpha^2 \end{bmatrix} \begin{bmatrix} h_N(0) \\
h_N(1) \\
h_N(2) \\
\vdots \\
h_N(N-1) \end{bmatrix} = \begin{bmatrix} 1 \\
0 \\
0 \\
\vdots \\
0 \end{bmatrix} \quad (9.22)$$

Solving these equations for arbitrary $\alpha$ and $N$ may be accomplished as follows. For $n = 1, 2, \ldots, N - 2$ these equations may be represented by the homogeneous difference equation,

$$-\alpha h_N(n - 1) + (1 + \alpha^2) h_N(n) - \alpha h_N(n + 1) = 0$$

The general solution to this equation is of the form

$$h_N(n) = c_1 \alpha^n + c_2 \alpha^{-n} \quad (9.23)$$

where $c_1$ and $c_2$ are constants that are determined by the boundary conditions at $n = 0$ and $n = N - 1$ [the first and last equations in Eq. (9.22)]:

$$\begin{align*}
(1 + \alpha^2) h_N(0) - \alpha h_N(1) &= 1 \\
-\alpha h_N(N - 2) + (1 + \alpha^2) h_N(N - 1) &= 0
\end{align*} \quad (9.24)$$

Substituting Eq. (9.23) into Eq. (9.24), we have

$$\begin{align*}
(1 + \alpha^2)[c_1 + c_2] - \alpha[c_1 \alpha + c_2 \alpha^{-1}] &= 1 \\
-\alpha[c_1 \alpha^{N-2} + c_2 \alpha^{-(N-2)}] + (1 + \alpha^2)[c_1 \alpha^{N-1} + c_2 \alpha^{-(N-1)}] &= 0
\end{align*}$$
which, after canceling common terms, may be simplified to
\[ c_1 + \alpha^2 c_2 = 1 \]
\[ \alpha^{N+1} c_1 + \alpha^{-(N-1)} c_2 = 0 \]

or
\[
\begin{bmatrix}
1 & \alpha^2 \\
\alpha^{N+1} & \alpha^{-(N-1)}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

The solution for \( c_1 \) and \( c_2 \) is
\[
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \frac{1}{\alpha^{-(N-1)} - \alpha^{N+3}}
\begin{bmatrix}
\alpha^{-(N-1)} \\
-\alpha^{N+1}
\end{bmatrix}
\]

Therefore, \( h_N(n) \) is
\[
h_N(n) = \begin{cases} 
\frac{\alpha^{n-N} - \alpha^{n-N-n}}{\alpha^{n-N} - \alpha^{n-N+2}} & 0 \leq n \leq N - 1 \\
0 & \text{else}
\end{cases}
\]

Let us now look at what happens asymptotically as \( N \to \infty \). If \( |\alpha| < 1 \),
\[
\lim_{N \to \infty} h_N(n) = \frac{\alpha^{-N}}{\alpha^{-N}} = \alpha^n \quad n \geq 0
\]

which is the inverse filter, that is,
\[
\lim_{N \to \infty} h_N(n) = \alpha^n u(n) = g^{-1}(n)
\]

and
\[
\lim_{N \to \infty} H_N(z) = \frac{1}{1 - \alpha z^{-1}}
\]

However, if \( |\alpha| > 1 \),
\[
\lim_{N \to \infty} h_N(n) = \frac{\alpha^{N-n}}{\alpha^{N+2}} = \alpha^{-n-2} \quad n \geq 0
\]

and
\[
\lim_{N \to \infty} H_N(z) = \frac{\alpha^{-2}}{1 - \alpha^{-1} z^{-1}}
\]

which is not the inverse filter. Note that although \( \hat{h}(n) = h_N(n) \ast g(n) \) does not converge to \( \delta(n) \) as \( N \to \infty \), taking the limit of \( \hat{D}_N(z) \) as \( N \to \infty \), we have
\[
\lim_{N \to \infty} \hat{D}_N(z) = \lim_{N \to \infty} \hat{H}_N(z)G(z) = \frac{1}{\alpha} \left( \frac{1 - \alpha z^{-1}}{\alpha - \alpha z^{-1}} \right)
\]

which is an all-pass filter, that is,
\[
|\hat{D}_N(e^{j\omega})| = \frac{1}{\alpha}
\]

9.53 The first five samples of the unit sample response of a causal filter are
\[
h(0) = 3 \quad h(1) = -1 \quad h(2) = 1 \quad h(3) = 2 \quad h(4) = 0
\]

If it is known that the system function has two zeros and two poles, determine whether or not the filter is stable.

The system function of this filter has the form
\[
H(z) = \frac{b(0) + b(1)z^{-1} + b(2)z^{-2}}{1 + a(1)z^{-1} + a(2)z^{-2}}
\]
To determine whether or not this system is stable, it is necessary to find the denominator polynomial,

\[ A(z) = 1 + a(1)z^{-1} + a(2)z^{-2} \]

and check to see whether or not the roots of \( A(z) \) lie inside the unit circle. Given that \( H(z) \) has two poles and two zeros, we may use the Padé approximation method to find the denominator coefficients:

\[
\begin{bmatrix}
  h(0) & 0 & 0 \\
  h(1) & h(0) & 0 \\
  h(2) & h(1) & h(0) \\
  h(3) & h(2) & h(1) \\
  h(4) & h(3) & h(2)
\end{bmatrix}
\begin{bmatrix}
  1 \\
  a(1) \\
  a(2)
\end{bmatrix}
= 
\begin{bmatrix}
  h(0) \\
  h(1) \\
  h(2) \\
  0 \\
  0
\end{bmatrix}
\]

Using the last two equations, we have

\[
\begin{bmatrix}
  h(3) & h(2) & h(1) \\
  h(4) & h(3) & h(2)
\end{bmatrix}
\begin{bmatrix}
  1 \\
  a(1) \\
  a(2)
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

which become

\[
\begin{bmatrix}
  h(2) & h(1) \\
  h(3) & h(2)
\end{bmatrix}
\begin{bmatrix}
  a(1) \\
  a(2)
\end{bmatrix}
= 
\begin{bmatrix}
  -h(3) \\
  -h(4)
\end{bmatrix}
\]

Substituting the given values for \( h(n) \), we have

\[
\begin{bmatrix}
  1 & -1 \\
  2 & 1
\end{bmatrix}
\begin{bmatrix}
  a(1) \\
  a(2)
\end{bmatrix}
= 
\begin{bmatrix}
  2 \\
  0
\end{bmatrix}
\]

The solution is

\[ a(1) = -\frac{2}{3} \quad a(2) = \frac{4}{3} \]

and the denominator polynomial is

\[ A(z) = 1 - \frac{2}{3}z^{-1} + \frac{4}{3}z^{-2} \]

Because the roots of this polynomial are not inside the unit circle, the filter is unstable.

**Supplementary Problems**

**FIR Filter Design**

9.54 What type of window(s) may be used to design a low-pass filter with a passband cutoff frequency \( \omega_p = 0.35\pi \), a transition width \( \Delta\omega = 0.025\pi \), and a maximum stopband deviation of \( \delta_s = 0.003 \)?

9.55 Use the window design method to design a minimum-order low-pass filter with a passband cutoff frequency \( \omega_s = 0.45\pi \), a stopband cutoff frequency \( \omega_s = 0.5\pi \), and a maximum stopband deviation \( \delta_s = 0.005 \).

9.56 We would like to design a bandstop filter to satisfy the following specifications:

\[
\begin{align*}
0.95 &\leq |H(e^{j\omega})| \leq 1.05 & 0 \leq \omega \leq 0.3\pi \\
|H(e^{j\omega})| &< 0.01 & 0.35\pi \leq \omega \leq 0.8\pi \\
0.95 &\leq |H(e^{j\omega})| \leq 1.05 & 0.85\pi \leq \omega \leq \pi
\end{align*}
\]

(a) What weighting function \( W(e^{j\omega}) \) should be used to design this filter?
9.57 Suppose that we would like to design a low-pass filter of order \( N = 128 \) with a passband cutoff frequency \( \omega_p = 0.48\pi \) and a stopband cutoff frequency of \( \omega_s = 0.52\pi \).

(a) Find the approximate passband and stopband ripple if we were to use a Kaiser window design.
(b) If an equiripple filter were designed so that it had a passband ripple equal to that of the Kaiser window design found in part (a), how small would the stopband ripple be?

9.58 We would like to design an equiripple low-pass filter of order \( N = 30 \). For a type I filter of order \( N \), what is the minimum number of alternations that this filter may have, and what is the maximum number?

9.59 For a low-pass filter with \( \delta_p = \delta_s \), what is the difference in the stopband attenuation in decibels between a Kaiser window design and an equiripple filter if both filters have the same transition width?

IIR Filter Design

9.60 Find the minimum order and the 3-dB cutoff frequency of a continuous-time Butterworth filter that will satisfy the following frequency response constraints:

\[ |H_a(j\Omega)| = 0.95 \quad \Omega = 16,000\pi \]
\[ |H_a(j\Omega)| \leq 0.1 \quad \Omega > 24,000\pi \]

9.61 Use the bilinear transformation to design a first-order low-pass Butterworth filter that has a 3-dB cutoff frequency \( \omega_c = 0.5\pi \).

9.62 Use the bilinear transformation to design a second-order bandpass Butterworth filter that has 3-dB cutoff frequencies \( \omega_0 = 0.4\pi \) and \( \omega_a = 0.6\pi \).

9.63 If the specifications for an analog low-pass filter are to have a 1-dB cutoff frequency of 1 kHz and a maximum stopband ripple \( \delta_s = 0.01 \) for \( |f| > 5 \) kHz, determine the required filter order for the following:
(a) Butterworth filter
(b) Type I Chebyshev filter
(c) Type II Chebyshev filter
(d) Elliptic filter

9.64 Let \( H_a(j\Omega) \) be an analog filter with
\[ H_a(j\Omega)|_{\Omega=0} = 1 \]

(a) If a discrete-time filter is designed using the impulse invariance method, is it necessarily true that
\[ H(e^{j\omega})|_{\omega=0} = 1 \]
(b) Repeat part (a) for the bilinear transformation.

9.65 Consider a causal and stable continuous-time filter that has a system function
\[ H_a(s) = \frac{s + 1}{(s + 2)^2} \]
If a discrete-time filter is designed using impulse invariance with \( T_c = 1 \), find \( H(z) \).

9.66 The system function of a digital filter is
\[ H(z) = \frac{2}{1 - 0.5z^{-1}} - \frac{1}{1 - 0.25z^{-1}} \]
(a) Assuming that this filter was designed using impulse invariance with $T_s = 2$, find the system function of two different analog filters that could have been the analog filter prototype.

(b) If this filter was designed using the bilinear transformation with $T_s = 2$, find the analog filter that was used as the prototype.

9.67 Determine the characteristics of the $s$-plane-to-$z$-plane mapping
\[
s = \frac{1 - z^{-2}}{1 + z^{-2}}
\]

9.68 The system function of an analog filter $H_a(s)$ may be expressed as a parallel connection of two lower-order systems
\[
H_a(s) = H_{a1}(s) + H_{a2}(s)
\]
If $H_a(s)$, $H_{a1}(s)$, and $H_{a2}(s)$ are mapped into digital filters using the impulse invariance technique, will it be true that
\[
H(z) = H_1(z) + H_2(z)
\]
What about with the bilinear transformation?

9.69 If an analog filter has an equiripple passband, will the digital filter designed using the impulse invariance method have an equiripple passband? Will it have an equiripple passband if the bilinear transformation is used?

9.70 Can an analog allpass filter be mapped to a digital allpass filter using the bilinear transformation?

9.71 An IIR low-pass digital filter is to be designed to meet the following specifications:
- Passband cutoff frequency of $0.22\pi$ with a passband ripple less than 0.01
- Stopband cutoff frequency of $0.24\pi$ with a stopband attenuation greater than 40 dB
(a) Determine the filter order required to meet these specifications if a digital Butterworth filter is designed using the bilinear transformation.
(b) Repeat for a digital Chebyshev filter.
(c) Compare the number of multiplications required to compute each output value using these filters, and compare them to an equiripple linear phase filter.

Least-Squares Filter Design

9.72 Suppose that the desired unit sample response of a linear shift-invariant system is
\[
h_d(n) = \delta(n) + 2\left(\frac{1}{2}\right)^n a(n - 1)
\]
Use the Padé approximation method to find the parameters of a filter with a system function
\[
H(z) = \frac{b(0) + b(1)z^{-1}}{1 + a(1)z^{-1}}
\]
that approximates this unit sample response.

9.73 The first five samples of the unit sample response of a causal filter are
\[
h(0) = 0.2000 \quad h(1) = 0.7560 \quad h(2) = 1.0737 \quad h(3) = -0.8410 \quad h(4) = -0.6739
\]
If it is known that the system function has two zeros and two poles, determine whether or not the filter is stable.
Answers to Supplementary Problems

9.54 A Hamming or a Blackman window or a Kaiser window with $\beta = 4.6$.

9.55 \( h(n) = w(n)h_b(n) \), where \( w(n) \) is a Kaiser window with $\beta = 4.09$ and $N = 107$, and

\[
h_b(n) = \frac{\sin[0.475\pi(n - 53.5)]}{(n - 53.5)\pi}
\]

9.56 (a) \( W(e^{j\omega}) = \begin{cases} 
1 & 0 \leq \omega \leq 0.3\pi \\
5 & 0.35\pi \leq \omega \leq 0.8\pi \\
1 & 0.85\pi \leq \omega \leq \pi
\end{cases} \)

(b) The minimum is 66 and the maximum is 69.

9.57 (a) \( \delta_o \approx \delta_i \approx 0.0058 \). (b) \( \delta_i \approx 0.0016 \).

9.58 The minimum number is 17 and the maximum is 18.

9.59 5 dB.

9.60 \( N = 9 \) and \( \Omega_c = 17.342\pi \).

9.61 \( H(z) = \frac{1}{2}(1 + z^{-1}) \).

9.62 \( H(z) = \frac{0.65(1 - z^{-2})}{2.65 + 1.35z^{-2}} \).

9.63 (a) \( N = 4 \). (b) \( N = 3 \). (c) \( N = 3 \). (d) \( N = 3 \).

9.64 (a) No. (b) Yes.

9.65 \( H(z) = \frac{1 - 2e^{-2z^{-1}}}{(1 - e^{-2z^{-1}})^2} \).

9.66 (a) One possible filter has a system function

\[
H_o(s) = \frac{2}{s - \frac{1}{2}\ln(0.5)} - \frac{1}{s - \frac{1}{2}\ln(0.25)}
\]

and another is

\[
H_o(s) = \frac{2}{s - \frac{1}{2}\ln(0.5) + j\pi} - \frac{1}{s - \frac{1}{2}\ln(0.25) + j\pi}
\]

Note, however, that the second filter has a complex-valued impulse response.

(b) This filter is unique and has a system function

\[
H_o(s) = \frac{4(1 + s)}{1 + 3s} - \frac{4(1 + s)}{3 + 5s}
\]

9.67 This is a cascade of two mappings. The first is the bilinear transformation, and the second is the mapping \( z \to z^2 \), which compresses the frequency axis by a factor of 2. Thus, a low-pass filter is mapped into a bandstop filter, and a high-pass filter is mapped into a bandpass filter.

9.68 True for both methods.

9.69 The digital filter will have an equiripple passband with the bilinear transformation but not with the impulse invariance method.
9.70 Yes.

9.71 (a) Butterworth filter order is \( N = 69 \).
(b) Chebyshev filter order is \( N = 17 \).
(c) For an equiripple filter, we require \( N = 185 \), which requires 185 delays. In addition, 93 multiplications are needed to evaluate each value of \( y(n) \). The Butterworth and Chebyshev filters require 69 and 17 delays, respectively, and approximately twice this number of multiplications to evaluate each value of \( y(n) \).

9.72 Padé gives \( b(0) = 1, b(1) = 0.5, \) and \( a(1) = -0.5 \), or

\[
H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.5z^{-1}}
\]

9.73 Padé’s method with \( p = q = 2 \) gives

\[
H(z) = \frac{0.2 + 0.8z^{-1} + 1.4z^{-2}}{1 + 0.22z^{-1} + 0.8z^{-2}}
\]

Because the roots of the denominator lie inside the unit circle, this filter is stable.