2 Internal Fluid Flow

2.1 Definitions

**Fluid Dynamics**

The study of fluids in motion.

**Static Pressure**

The pressure at a given point exerted by the static head of the fluid present directly above that point. Static pressure is related to motion on a molecular scale.

**Dynamic or Velocity Pressure**

Dynamic pressure is related to fluid motion on a large scale i.e. fluid velocity.
Stagnation Pressure Total Pressure

The sum of the static pressure plus the dynamic pressure of a fluid at a point.

Streamline

An imaginary line in a moving fluid across which, at any instant, no fluid is flowing. ie it indicates the instantaneous direction of the flow.

Stream tube

A ‘bundle’ of neighbouring streamlines may be imagined to form a stream tube (not necessarily circular) through which the fluid flows.
Control volume

A fixed volume in space through which a fluid is continuously flowing. The boundary of a control volume is termed the **control surface**. The size and shape is entirely arbitrary and normally chosen such that it encloses part of the flow of particular interest.

Classification of Fluid behaviour

a) Steady or unsteady
   
   A flow is termed steady if its properties do not vary with time.  
   A flow is termed unsteady if properties at a given point vary with time.  
   Quasi-steady flow is essentially unsteady but its properties change sufficiently slowly with respect to time, at a given point, that a series of steady state solutions will approximately represent the flow.

b) Uniform or Non-uniform
   
   A uniform flow is one in which properties do not vary from point to point over a given cross-section.  
   Non-uniform flow has its properties changing with respect to space in a given cross-section.

c) One-dimensional or Multi-dimensional
   
   One-dimensional flow, is one in which the direction and magnitude of the velocity at all points are identical.  
   Variation of velocity in other directions is so small that they can be neglected. eg. flow of water in small bore pipe at low flow rates.  
   Two-dimensional flow is one in which the velocity has two main components.  
   Three-dimensional flow is one in which the flow velocity has significant components in all three directions.

d) Viscid or Inviscid
   
   This sometimes distinguished as Viscid and inviscid flow in relation to the viscous forces whether they are neglected or taken into account

e) Compressible or Incompressible
   
   If the changes in density are relatively small, the fluid is said to be incompressible. If the changes in density are appreciable, in case of the fluid being subjected to relatively high pressures, the fluid has to be treated as Compressible.

f) Ideal or Real
   
   An ideal fluid is both inviscid and incompressible. This definition is useful in forming analytical solution to fluid flow problems.

   Fluids in reality are viscous and compressible. Thus, the effect of compressibility and viscosity must be considered for accurate analysis. It must be stressed that in most common engineering applications at standard pressure and temperature, water can be assumed incompressible and inviscid. The assumption of ideal fluid can help to formulate a solution, an approximate solution, still better than no solution.
2.2 Conservation of Mass

The continuity equation applies the principle of conservation of mass to fluid flow. Consider a fluid flowing through a fixed volume tank having one inlet and one outlet as shown below.

![Figure 2.2 conservation of mass](image)

If the flow is steady i.e. no accumulation of fluid within the tank, then the rate of fluid flow at entry must be equal to the rate of fluid flow at exit for mass conservation. If, at entry (or exit) having a cross-sectional area $A$ (m$^2$), a fluid parcel travels a distance $dL$ in time $dt$, then the volume flow rate ($V$, m$^3$/s) is given by:

$$V = \frac{A \cdot dL}{\Delta t}$$

but since $dL/\Delta t$ is the fluid velocity ($v$, m/s) we can write:

$$Q = V \times A$$

The mass flow rate ($m$, kg/s) is given by the product of density and volume flow rate

i.e.  

$$m = \rho \cdot Q = \rho \cdot V \cdot A$$

Between two points in flowing fluid for mass conservation we can write:

$$m_1 = m_2$$

or

$$\rho_1 \cdot V_1 \cdot A_1 = \rho_2 \cdot V_2 \cdot A_2$$

(2.1)

If the fluid is incompressible i.e. $\rho_1 = \rho_2$ then:

$$V_1 \cdot A_1 = V_2 \cdot A_2$$

(2.2)

Hence an incompressible flow in a constant cross-section will have a constant velocity. For branched systems the continuity equation implies that the sum of the incoming fluid mass (or volume) flow rates must equal the sum of the outgoing mass (or volume) flow rates.

**Worked Example 2.1**

Air enters a compressor with a density of 1.2 kg/m$^3$ at a mean velocity of 4 m/s in the 6 cm x 6 cm square inlet duct. Air is discharged from the compressor with a mean velocity of 3 m/s in a 5 cm diameter circular pipe. Determine the mass flow rate and the density at outlet.
Solution:

Given: \( \rho_1 = 1.2 \text{ kg/m}^3, \ V_1 = 4 \text{ m/s}, \ V_2 = 3 \text{ m/s} \)

\[
A_1 = 0.06 \times 0.06 = 0.0036 \text{ m}^2
\]

\[
A_2 = \frac{\pi D^2}{4} = \frac{3.142 \times 0.05^2}{4} = 0.00196 \text{ m}^2
\]

The mass flow rate is:

\[
m = \rho_1 A_1 V_1
\]

\[
= 1.2 \times 0.0036 \times 4
\]

\[
= 17.28 \times 10^{-3} \text{ kg/s}
\]

Conservation of mass between sections 1 and 2 implies that:

\[
\rho_1 A_1 V_1 = \rho_2 A_2 V_2
\]
Hence the density at section 2 is calculated:

\[
\rho_2 = \frac{m}{A_2 x V_2} = \frac{17.28 \times 10^{-3}}{0.00196 \times 3} = 2.933 \text{ kg/m}^3
\]

### 2.3 Conservation of Energy

There are three forms of non-thermal energy for a fluid at any given point:-

- **The kinetic energy** due to the motion of the fluid.
- **The potential energy** due to the positional elevation above a datum.
- **The pressure energy**, due to the absolute pressure of the fluid at that point.

Conservation of energy necessitates that the total energy of the fluid remains constant, however, there can be transformation from one form to another.

If all energy terms are written in the form of the head (potential energy), ie in metres of the fluid, then:

- \(\frac{p}{\rho g}\) represents the pressure head (sometimes known as ‘flow work’)
- \(\frac{V^2}{2g}\) represents the velocity head (also known as kinetic energy)

The energy conservation, thus, implies that between any two points in a fluid

\[
\left( \frac{p}{\rho g} + \frac{V^2}{2g} + z \right)_1 = \left( \frac{p}{\rho g} + \frac{V^2}{2g} + z \right)_2
\]

This equation is known as the **Bernoulli equation** and is valid if the two points of interest 1 & 2 are very close to each other and there is no loss of energy.

In a real situation, the flow will suffer a loss of energy due to friction and obstruction between stations 1 & 2, hence

\[
\left( \frac{p}{\rho g} + \frac{V^2}{2g} + z \right)_1 = \left( \frac{p}{\rho g} + \frac{V^2}{2g} + z \right)_2 + h_L
\]

where \(h_L\) is the loss of energy between the two stations.

When the flow between stations 1 & 2 is caused by a pump situated between the two stations, the energy equation becomes:

\[
\left( \frac{p}{\rho g} + \frac{V^2}{2g} + z \right)_1 + h_p = \left( \frac{p}{\rho g} + \frac{V^2}{2g} + z \right)_2 + h_L
\]

Where \(h_p\) is the head gain due to the pump.
Worked Example 2.2

A jet of water of 20 mm in diameter exits a nozzle directed vertically upwards at a velocity of 10 m/s. Assuming the jet retains a circular cross-section, determine the diameter (m) of the jet at a point 4.5 m above the nozzle exit. Take $\rho_{\text{water}} = 1000 \text{ kg/m}^3$.

Solution:

Bernoulli equation: 
\[
\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2
\]

Given: $v, z_1 = 0$ (Datum) $z_2 = 4.5 \text{ m}$. $p_1 = p_2$ (both atmospheric). The energy equation reduces to:

\[
\frac{V_2^2}{2g} = \frac{V_1^2}{2g} + z_1 - z_2
\]

\[
\therefore \quad V_2 = \sqrt{V_1^2 - 2gh} = \sqrt{10^2 - 19.62 \times 4.5} = 3.42 \text{ m/s}
\]

From continuity equation: 
\[
A_2 = \frac{A_1 x V_1}{V_2} = \frac{\pi x 0.01^2 \times 10}{3.42} = 9.18 \times 10^{-4} \text{ m}^2
\]
Hence

\[ A_2 = \frac{\pi D_2^2}{4} \]

\[ D_2 = \sqrt{\frac{4 \times 9.18 \times 10^{-4}}{3.14}} = 0.034m \]

2.4 Flow Measurement

There are a large number of devices for measuring fluid flow rates to suit different applications. Three of the most commonly encountered restriction methods will be presented here.

Restriction methods of fluid flow are based on the acceleration or deceleration of the fluid through some kind of nozzle, throat or vena contracta.

Theoretical analysis applies the continuity and Bernoulli equations to an ideal fluid flow between points 1 and 2 thus:-

Start with Bernoulli equation:

\[ \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 \]

Rearranging

\[ p_1 - p_2 = \frac{D_2 v_2^2}{2} \left[ 1 - \left( \frac{V_1}{V_2} \right)^2 \right] \]
Then use the continuity equation \[ V_1 A_1 = V_2 A_2 \]

Therefore \( \frac{V_1}{V_2} = \left( \frac{A_2}{A_1} \right)^2 \)

Substituting into the rearranged Bernoulli equation and solving for \( V_2 \) we have:-

\[
V_2 = \frac{2(p_1 - p_2)}{\rho [1 - (A_2/A_1)^2]}
\]  

(2.4)

The theoretical volume flow rate is \( Q = A_2 V_2 \)

And the theoretical mass flow rate is \( \dot{m} = \rho A_2 V_2 \)

The above values are theoretical because ideal fluid flow conditions were assumed. Actual flow rate values are obtained by multiplying the theoretical values by a meter discharge coefficient \( C_d \) to account for frictional and obstruction losses encountered by the fluid in its passage through the meter. The energy losses manifest themselves as a greater pressure drop \( (P_1 - P_2) \) than that predicted by the theory.

It can be shown that \( V_a = C_d x \frac{2(p_1 - p_2)}{\rho [1 - (D_2 / D_1)^4]} \)  

(2.5)

(a) The Venturi meter

The Venturi meter has a converging section from the initial pipe diameter down to a throat, followed by a diverging section back to the original pipe diameter. See figure.

Figure 2.2 the Venturi tube

Differential pressure measurements are taken between the inlet (1) and throat (2) positions. The geometry of the meter is designed to minimize energy losses \( (C_d > 0.95) \).
(b) **The Orifice meter**

An orifice meter is a flat plate, with a hole which may be square edged or bevelled, inserted between two flanges in a pipe line. In this instance positions (1) and (2) are as shown below. Orifice plates have a simple construction and are therefore inexpensive but they suffer from high energy losses ($C_d = 0.6$).

![Image of Orifice meter](image)

**Figure 2.3** the Orifice meter

(c) **The Pitot-static tube**

A slender concentric tube arrangement, aligned with the flow, used to measure flow velocity by means of a pressure difference. See figure below. The outer tube is closed in the flow direction but has sidewall holes to enable the measurement of static pressure. The inner tube is open in the direction of the fluid flow and is thus experiencing the total (static + dynamic) pressure of the fluid flow. It is assumed that the fluid velocity is rapidly brought to zero upon entry to the inner tube with negligible friction ($C_d \sim 1$). The pressure difference between the tubes is applied to a U tube manometer which will therefore indicate the velocity pressure.

Start with Bernoulli equation:

\[
\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2
\]

Since the Pitot-static tube is mounted horizontally, the $z$-terms will cancel out, and the static end is motionless, i.e. $V_2 = 0$. It can be shown that the duct velocity $V_1$ is given by:-

\[
V_1 = \sqrt{\frac{2(P_1 - P_e)}{\rho_l}} = \sqrt{\frac{2 \rho_m \cdot g \cdot h}{\rho_l}} 
\]

(2.6)
Worked Example 2.3

A Venturi meter fitted in a 15 cm pipeline has a throat diameter of 7.5 cm. The pipe carries water, and a U-tube manometer mounted across the Venturi has a reading of 95.2 mm of mercury. Determine:

1. the pressure drop in Pascal's, indicated by the manometer
2. the ideal throat velocity (m/s)
3. the actual flow rate (l/s) if the meter $C_D$ is 0.975.
Solution:

(i) \[ p_1 - p_2 = \rho_m \times g \times h \]
\[ = 13600 \times 9.81 \times 0.0952 = 12701 \text{ Pa} \]

(ii) \[ v_2 = \sqrt{\frac{2(\rho_1 - \rho_2)}{\rho[1 - (A_2 / A_1)^2]}} = \sqrt{\frac{2(12701)}{1000[1 - (0.0629)]}} = 5.206 \text{ m} / \text{s} \]

(iii) \[ Q = C_d v_2 A_2 \]
\[ = 0.975 \times 5.206 \times 0.00441 \]
\[ = 0.0224 \text{ m}^3/\text{s} = 22.4 \text{ l/s} \]

2.5 Flow Regimes

Consider the variation in velocity across the cross-section of a pipe containing a fluid in motion. There is no motion of fluid in direct contact with the pipe wall, and the velocity of the fluid stream increases in a direction away from the walls of the pipe. In 1839, Hagen (USA) observed that the fluid moves in layers with a velocity gradient. He observed that the velocity gradient in a circular pipe follows a parabolic law, at low flow rates. This type of flow is termed **LAMINAR**.

When the flow rate of the fluid stream is high, the velocity distribution had a much flatter shape and this type of flow is known as **TURBULENT**.

The average velocity producing turbulent flow is greater than that for a laminar flow of a given fluid in a given duct.

For both flows, the build-up of velocity is along the radius of the pipe, and the maximum velocity occurs at the centre line.
Osborne Reynolds demonstrated experimentally in 1883 (Manchester) that under laminar flow, the fluid streamlines remain parallel. This was shown with the aid of a dye filament injected in the flow which remained intact at low flow velocities in the tube. As the flow velocity was increased (via a control valve), a point was reached at which the dye filament at first began to oscillate then broke up so that the colour was diffused over the whole cross-section indicating that particles of fluid no longer moved in an orderly manner but occupied different relative positions in successive cross-sections downstream.

Reynolds also found that it was not only the average pipe velocity \( V \) which determined whether the flow was laminar or turbulent, but that the density \( (\rho) \) and viscosity \( (\mu) \) of the fluid and the pipe diameter \( (D) \), also determined the flow regime. He proposed that the criterion which determined the type of regime was the dimensionless group \( (\rho vD/\mu) \). This group has been named the **Reynolds number** \((\text{Re})\) as a tribute to his contribution to Fluid Mechanics.

\[
\text{Re} = \frac{\rho V D}{\mu}
\]

Based on Reynolds number the flow can be distinguished into three regimes for pipe flow:

- Laminar if \( \text{Re} < 2000 \)
- Transitional if \( 2000 < \text{Re} < 4000 \)
- Turbulent if \( \text{Re} > 4000 \)
  
  \( \text{Re} = 2000, 4000 \) are the lower and upper critical values.

### 2.6 Darcy Formula

Consider a duct of length \( L \), cross-sectional area \( A_c \), surface area \( A_s \), in which a fluid of density \( \rho \), is flowing at mean velocity \( V \). The forces acting on a segment of the duct are that due to pressure difference and that due to friction at the walls in contact with the fluid.

If the acceleration of the fluid is zero, the net forces acting on the element must be zero, hence

\[
\begin{align*}
\sum F &= 0 \\
(P_1 - P_2). A_c - (f \rho V^2/2). A_s &= 0
\end{align*}
\]

According to Newton’s Second Law of Motion for a constant velocity flow:

\[
\sum F = 0
\]

The force due to pressure on either side of the section is equal to the friction force resisting the flow:

\[
(P_1 - P_2). A_c = (f \rho V^2/2). A_s
\]
Where the pressures act normal to the flow direction on the area of cross-section $A_c$, and the frictional force acts on the circumferential wall area $A_s$, separating the fluid and the pipe's surface.

Let $h_f$ denote the head lost (m) due to friction over a duct length $L$,

$$p_1 - p_2 = \rho g h_f$$

Substituting we get

$$h_f = f \cdot (A_s/A_c) \cdot \frac{V^2}{2g}$$

For a pipe $A_s/A_c = \pi D L / \pi D^2/4 = 4L/D$

$$h_f = (4fL/D) . \frac{V^2}{2g} \quad (2.8)$$

This is known as Darcy formula.

### 2.7 The Friction factor and Moody diagram

The value of the friction factor ($f$) depends mainly on two parameters namely the value of the Reynolds number and the surface roughness.
For *laminar* flow (i.e. $Re < 2000$), the value of the friction factor is given by the following equation *irrespective of the nature of the surface*:

\[ f = \frac{16}{Re^{0.33}} \quad (2.9) \]

While for a *smooth* pipe with *turbulent* (i.e. $Re > 4000$) flow, the friction factor is given by:

\[ f = \frac{0.079}{Re} \quad \text{(Blasius equation)} \quad (2.10) \]

For $Re > 2000$ and $Re < 4000$, this region is known as the critical zone and the value of the friction factor is uncertain and not quoted on the Moody diagram (Figure 2.5).

In the turbulent zone, if the surface of the pipe is not perfectly smooth, then the value of the friction factor has to be determined from the *Moody diagram*.

The *relative roughness* ($k/d$) is the ratio of the average height of the surface projections on the inside of the pipe ($k$) to the pipe diameter ($D$). In common with Reynolds number and friction factor this parameter is dimensionless. Values of $k$ are tabled on the Moody chart for a sample of materials.
Figure 2.5 the Moody Chart/diagram
Worked Example 2.4

Water flows in a 40mm diameter commercial steel pipe ($k = 0.045 \times 10^{-3}$ m) at a rate of 1 litre/s. Determine the friction factor and head loss per metre length of pipe using:

1. The Moody diagram
2. Smooth pipe formulae. Compare the results.

Take: $\rho = 1000 \text{ kg/m}^3$, $\mu = 1 \times 10^{-3} \text{ kg/ms}$

Solution:

$V = \frac{Q}{A} = \frac{0.001}{1.256 \times 10^{-3}} = 0.796 \text{ m}^2$

$Re = \frac{\rho V D}{\mu}$

$= 1000 \times 0.796 \times 0.04/ 1 \times 10^{-3} = 31840 \text{ i.e. turbulent}$

1. Moody diagram
$k/D = \frac{0.045 \times 10^{-3}/0.04}{0.0011}$

From intersection of $k/D$ and $Re$ values on Moody diagram read off $f = 0.0065$
Therefore

\[ h_r = 4x f x \frac{L v^2}{D 2g} \]

\[ = 4 \times 0.0065 \times 1 \times 0.796^2 \]

\[ = 0.0209 \text{ m/m pipe} \]

2. Using Blasius equation for smooth pipe:

\[ f = \frac{0.079}{Re^{0.25}} \]

\[ = \frac{0.079}{(31840)^{0.25}} = 0.0059 \]

\[ h_r = 4 \times 0.0059 \times 1 \times 0.796^2 \]

\[ = 0.02 \text{ m/m pipe} \]

i.e. 9% less than Moody.

Note that if the pipe is assumed smooth, the friction factor from the Moody diagram would be \( f = 0.0058 \) which is closer to the Blasius value.

### 2.8 Flow Obstruction Losses

When a pipe changes direction, changes diameter or has a valve or other fittings there will be a loss of energy due to the disturbance in flow. This loss of energy \( (h_o) \) is usually expressed by:

\[ h_o = K \frac{V^2}{2g} \quad (2.11) \]

Where \( V \) is the mean velocity at entry to the fitting and \( K \) is an empirically determined factor. Typical values of \( K \) for different fittings are given in the table below:
### Table 2.1: Obstruction Losses in Flow Systems

<table>
<thead>
<tr>
<th>Obstruction</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>tank exit</td>
<td>0.5</td>
</tr>
<tr>
<td>tank entry</td>
<td>1.0</td>
</tr>
<tr>
<td>smooth bend</td>
<td>0.30</td>
</tr>
<tr>
<td>Mitre bend</td>
<td>1.1</td>
</tr>
<tr>
<td>Mitre bend with guide vanes</td>
<td>0.2</td>
</tr>
<tr>
<td>90 degree elbow</td>
<td>0.9</td>
</tr>
<tr>
<td>45 degree elbow</td>
<td>0.42</td>
</tr>
<tr>
<td>Standard T</td>
<td>1.8</td>
</tr>
<tr>
<td>Return bend</td>
<td>2.2</td>
</tr>
<tr>
<td>Strainer</td>
<td>2.0</td>
</tr>
<tr>
<td>Globe valve, wide open</td>
<td>10.0</td>
</tr>
<tr>
<td>Angle valve, wide open</td>
<td>5.0</td>
</tr>
<tr>
<td>Gate valve, wide open</td>
<td>0.19</td>
</tr>
<tr>
<td>(\frac{1}{4}) open</td>
<td>1.15</td>
</tr>
<tr>
<td>(\frac{1}{2}) open</td>
<td>5.6</td>
</tr>
<tr>
<td>(\frac{3}{4}) open</td>
<td>24.0</td>
</tr>
<tr>
<td>Sudden enlargement</td>
<td>0.10</td>
</tr>
<tr>
<td>Conical enlargement: (6^\circ) (total included angle)</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>0.55</td>
</tr>
<tr>
<td>Sudden contractions: area ratio 0.2</td>
<td>0.41</td>
</tr>
<tr>
<td>(\frac{A_2}{A_1}) 0.4</td>
<td>0.30</td>
</tr>
<tr>
<td>0.6</td>
<td>0.18</td>
</tr>
<tr>
<td>0.8</td>
<td>0.06</td>
</tr>
</tbody>
</table>

#### 2.9 Fluid Power

The fluid power available at a given point for a fluid is defined as the product of mass, acceleration due to gravity and the fluid head, and since the mass flow rate is defined as the volume flow rate multiplied by the fluid density, the Fluid power therefore can be expressed as:

\[
P = \rho \cdot g \cdot Q \cdot h_{\text{tot}}
\]  

(2.12a)

For a **pump**, \(h_{\text{tot}}\) represents the head required to overcome pipe friction (\(h_f\)), obstruction losses (\(h_o\)) and to raise the fluid to any elevation required (\(h_z\)).
\[ h_{tot} = h_z + h_f + h_o \]  \hspace{1cm} (2.13a)

Note: If the delivery tank operates at pressure in excess of the supply tank an additional term \( h_p \) must be added to the required head equation as this pressure rise must also be supplied by the pump.

If the pump efficiency \( \eta_p \) is introduced, the actual pump head requirement is:

\[ P = \frac{\rho \cdot g \cdot Q \cdot h_{tot}}{\eta_p} \]  \hspace{1cm} (2.12b)

For a turbine with efficiency \( \eta_t \), the power output is given by:

\[ P = \frac{\rho \cdot g \cdot Q \cdot h_{tot} \cdot \eta_t}{(2.12c)} \]

Where \( h_{tot} = h_z - (h_f + h_o) \) \hspace{1cm} (2.13b)

**Worked Example 2.5**

Determine the input power to an electric motor \( (\eta_m = 90\%) \) supplying a pump \( (\eta_p = 80\%) \) delivering 50 l/s of water \( (\rho = 1000 \text{ kg/m}^3, \mu = 0.001 \text{ kg/ms}) \) from tank1 to tank 2 as shown below if the pipeline length is 200m long, of 150 mm diameter galvanised steel \( (\text{assumed surface roughness } k=0.15\text{mm})\).
Solution:

\[ V = \frac{Q}{A} = 0.05 / (\pi \times 0.15^2 / 4) = 2.83 \text{ m/s} \]

\[ \text{Re} = \frac{\rho V D}{\mu} = 1000 \times 2.83 \times 0.15 / 0.001 = 4.244 \times 10^5 \]

\[ k/D = 0.15 / 150 = 0.001; \text{ From Moody diagram } f = 0.0051 \]

\[ h_f = \frac{4 f L V^2}{2g} = \frac{4 \times 0.0051 \times 200}{0.15} \times \frac{2.83^2}{2 	imes 9.81} = 11.1 \text{ m} \]

Three bends (each K=0.9), tank entry (k=0.5), exit loss (k=1) and one valve (k=5)

\[ h_o = \sum K_i \frac{V^2}{2g} = (3 \times 0.9 + 0.5 + 1 + 5) \times \frac{2.83^2}{2 	imes 9.81} = 3.75 \text{ m} \]

\[ h_{sys} = H_f + H_x + H_o = 11.1 + 3.75 + 100 = 114.85 \text{ m} \]

Input power

\[ P = \frac{\rho g Q H_{sys}}{\eta_m \eta_p} = \frac{1000 \times 9.81 \times 0.05 \times 114.85}{0.9 \times 0.8} = 78.3 \text{ kW} \]

2.10 Fluid Momentum

Knowledge of the forces exerted by moving fluids is important in the design of hydraulic machines and other constructions. The Continuity and Bernoulli relationships are not sufficient to solve forces acting on bodies in this case and the momentum principle derived from Newton's Laws of motion is also required.
Momentum is defined as the product of mass and velocity, and represents the energy of motion stored in the system. Momentum is a vector quantity and can only be defined by specifying its direction as well as magnitude.

**Newton's Second Law of Motion**

“The rate of change of momentum is proportional to the net force acting, and takes place in the direction of that force”.

\[
\text{i.e. } \sum F = \frac{dM}{dt}
\]

Since \( M = m \cdot V \)

\[
\sum F = \frac{dV}{dt} + V \frac{dm}{dt}
\]

**Newton's Third Law of Motion**

“‘To every action there is a reaction equal in magnitude and opposite in direction’.”

**Application of Momentum Equation**

Consider horizontal jet impinging a surface tangentially at a steady state.

Resolving horizontally we have for the x-component

\[- F_x = \dot{m}(V_2 \cos \alpha - V_1) \sin \alpha \frac{dm}{dt} = 0 \ (steady \ flow)\]

Resolving vertically we have for the y-component

\[ F_y = \dot{m}(V_2 \sin \alpha - 0) \]
The resultant force acting on the solid surface due to the jet is given by

$$F_R = \sqrt{F_x^2 + F_y^2}$$

acting through

$$\theta_R = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

Let $V_2 = kV_1$ where $k$ is 'Friction Coefficient' ($0<k<1$)

$$\therefore F_x = \rho A V_1^2 (1 - k \cos \theta) \quad \text{and} \quad F_y = \rho A V_1^2 k \sin \theta$$

If smooth, then $k = 1$ and

$$F_x = \rho A V_1^2 (1 - \cos \theta), \quad F_y = \rho A V_1^2 \sin \theta$$

Special cases:

The angle of the striking jet has a very important effect on the force, 3 different angles are illustrated below:
Worked Example 2.6

A jet of water having a diameter of 7.5 cm and a velocity of 30 m/s strikes a stationary flat plate at angle \( \theta = 30^\circ \) as shown below.

**Solution:**

\[ A = 0.00442 \text{ m}^2 \]

Smooth i.e. \( k = 1 \), \( V_1 = V_2 = 30 \text{ m/s} \)

\[ F_x = \rho AV_1^2 (1 - \cos \theta) \]
\[ = 1000 \times 0.00442 \times 30^2 (1 - \cos 30) \]
\[ = 533 \text{ N} \]
Flow forces on a Reducer Bend

The change of momentum of a fluid flowing through a pipe bend induces a force on the pipe.

The pressures are to be considered in this case since the reducer bend is part of flowing system which is not subjected to atmospheric conditions.

\[ F_x = \rho A_1 V_1^2 \sin \theta \]
\[ = 1000 \times 0.00442 \times 30^2 \times \sin 30 \]
\[ = 1989 \text{ N} \]

\[ F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{533^2 + 1989^2} = 2060 \text{ N} \]

\[ \theta_k = \tan^{-1}\left\{ \frac{F_y}{F_x} \right\} = \tan^{-1}\left\{ \frac{1989}{533} \right\} = 75^\circ \]

x - Momentum

\[ -F_x + p_1 A_1 - p_2 A_2 \cos \theta = \rho \hspace{1mm} \dot{V} \hspace{1mm} (V_2 \cos \theta - V_1) \]

y - Momentum

\[ -F_y + p_2 A_2 \sin \theta = \rho \hspace{1mm} Q \hspace{1mm} (-V_2 \sin \theta - 0) \]

The total force

\[ F = \sqrt{F_x^2 + F_y^2} \]

\[ = (\rho^2 Q^2 V_1^2 (1 - 2 \frac{A_1}{A_2} \cos q + \frac{A_1^2}{A_2^2}) + 2r \dot{V}^2 [p_1 + p_2 \left( -p_1 \frac{A_1}{A_2} + \text{etc.} \right)])^{1/2} \]

From continuity equation:

\[ V_2 = V_1 \times \frac{A_1}{A_2} \]

For \( A_1 = A_2 \) and \( \theta = 90^\circ \)
The force acting at an angle \[ \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) \] with the x-axis.

**Worked Example 2.7**

A bend in a horizontal pipeline reduces from 600 mm to 300 mm whilst being deflected through 60°. If the pressure at the larger section is 250 kPa, for a water flow rate of 800 l/s determine the magnitude and direction of the resulting force on the pipe.

Take \( \rho_{\text{water}} = 1000 \text{ kg/m}^3 \)
Solution:

From Continuity

\[ Q = V_1 A_1 \quad V_1 = 0.8/0.282 = 2.83 \text{ m/s} \]

\[ Q = V_2 A_2 \quad V_2 = 0.8/0.07 = 11.42 \text{ m/s} \]

From Bernoulli equation:

\[ P_2 = P_1 + \frac{1}{2} \rho (V_1^2 - V_2^2) = 250 \times 10^3 + 0.5 \times 1000 \times (2.83^2 - 11.42^2) = 188.8 \text{ kPa} \]

From Momentum equation:

\[ F_x = p_1 A_1 \cos \theta - p_2 A_2 \cos \theta - \rho Q (V_2 \cos \theta - V_1) \]

\[ = [250 \times 10^3 \times 0.282] - [188.8 \times 10^3 \times 0.07 \times 0.5] - [1000 \times 0.8((11.42 \times 0.5) - 2.83)] \]

\[ = 61589.4 \text{ N} \]

\[ F_y = p_2 A_2 \sin \theta - \rho Q (-V_2 \sin \theta - 0) \]

\[ = [188.796 \times 10^3 \times 0.07 \times 0.866] - [1000 \times 0.8(-9.889)] = 19356 \text{ N} \]

\[ F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{61589.4^2 + 19356^2} = 64.559 \times 10^3 \text{ N} \]

\[ \theta_R = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{19356}{61589}\right) = 17.44^\circ \]

Worked Example 2.8

A siphon has a uniform circular bore of 75 mm diameter and consists of a bent pipe with its crest 1.4 m above water level and a discharge to the atmosphere at a level 2 m below water level. Find the velocity of flow, the discharge and the absolute pressure at crest level if the atmospheric pressure is 98.1 kN/m². Neglect losses due to friction.
Solution:

Bernoulli equation between 1-3:
\[
\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_3}{\rho g} + \frac{v_3^2}{2g} + z_3
\]

\(z_1 = 2, z_3 = 0\) (level 3 is assumed datum). \(p_1 = p_3\) (both atmospheric). And \(V_1 = 0\)

The energy equation reduces to:

\[
0 + 0 + 2 = 0 + \frac{V_3^2}{2g}
\]

\[\text{hence} \rightarrow V_3 = \sqrt{2 \times 9.81 \times 2} = 6.264 \text{ m/s}\]

The flow rate is calculated:

\[
Q = AxV = \frac{\pi}{4} \times 0.075^2 \times 6.264 = 0.0277 m^3/s
\]

Applying Bernoulli equation between 1 and 2:

\[
\frac{98.1 \times 10^3}{1000 \times 9.81} + 0 + 2 = \frac{p_2}{\rho g} + \frac{6.264^2}{19.61} + 3.4
\]

\[\text{hence} \rightarrow P_2 = 49.050 \text{ kPa}\]
2.11 Tutorial Problems

2.1 A 20 mm diameter pipe forks, one branch being 10 mm in diameter and the other 15 mm in diameter. If the velocity in the 10 mm pipe is 0.3 m/s and that in the 15 mm pipe is 0.6 m/s, calculate the rate of flow in cm³/s and velocity in m/s in the 20 mm diameter pipe.

(129.6 cm³/s, 0.413 m/s)

2.2 Water at 36 m above sea level has a velocity of 18 m/s and a pressure of 350 kN/m². Determine the potential, kinetic and pressure energy of the water in metres of head. Also determine the total head.

Ans (35.68 m, 16.5 m, 36 m, 88.2 m)

2.3 The air supply to an engine on a test bed passes down a 180 mm diameter pipe fitted with an orifice plate 90 mm diameter. The pressure drop across the orifice is 80 mm of paraffin. The coefficient of discharge of the orifice is 0.62 and the densities of air and paraffin are 1.2 kg/m³ and 830 kg/m³ respectively. Calculate the mass flow rate of air to the engine.

Ans (0.16 kg/s)
2.4 Determine the pressure loss in a 100 m long, 10 mm diameter smooth pipe if the flow velocity is 1 m/s for:

a) air whose density 1.0 kg/m³ and dynamic viscosity $1 \times 10^{-5}$ Ns/m².

b) water whose density 1000 kg/m³ and dynamic viscosity $1 \times 10^{-3}$ Ns/m².

Ans: (320 N/m², 158 kN/m²).

2.5 Determine the input power to an electric motor ($\eta_m = 90\%$) supplying a pump ($\eta_p = 90\%$) delivering 50 l/s of water ($\rho = 1000$ kg/m³, $\mu = 0.001$ kg/ms) between two tanks with a difference in elevation of 50m if the pipeline length is 100m long in total of 150 mm diameter, assume a friction factor of 0.008 and neglect minor losses.

Ans: (33.6 kW).

2.6 A jet of water strikes a stationary flat plate “perpendicularly”, if the jet diameter is 7.5 cm and its velocity upon impact is 30 m/s, determine the magnitude and direction of the resultant force on the plate, neglect frictional effect and take water density as 1000 kg.m⁴.

Ans (3970 N)

2.7 A horizontally laid pipe carrying water has a sudden contraction in diameter from 0.4m to 0.2m respectively. The pressure across the reducer reads 300 kPa and 200 kPa respectively when the flow rate is 0.5 m³/s. Determine the force exerted on the section due to the flow, assuming that friction losses are negligible.

Ans: (25.5 kN).

2.8 A siphon has a uniform circular bore of 75 mm diameter and consists of a bent pipe with its crest 1.8 m above water level and a discharge to the atmosphere at a level 3.6 m below water level. Find the velocity of flow, the discharge and the absolute pressure at crest level if the atmospheric pressure is 98.1 kN/m². Neglect losses due to friction.

Ans (0.0371 m³/s, 45.1 kN/m²)